SIMILITUDE AND DIMENSIONAL ANALYSIS

It is usually impossible to determine all the essential facts for a given fluid flow by pure theory and hence dependence must often be placed upon experimental investigation. The number of tests to be made can be greatly reduced by;

(i) Systematic program based on dimensional analysis

(ii) Application of the laws of similitude or similarity

The laws of similitude make it possible to determine the performance of the **prototype**, which means the full-size device from tests made with **model**.

- ***Important hydraulic structures are now designed and built only after extensive model studies have been made.***
- Application of DIMENSIONAL ANALYSIS and hydraulic SIMILITUDE enable the engineer to **organize** and **simplify** the experiments and to **analyze** the results.

HYDRAULIC MODELS

Hydraulic models in general may be either (i) TRUE MODELS (ii) DISTORTED MODELS.

True models have all significant characteristics of the prototype reproduced to scale i.e. (geometrically similar) and satisfy design restrictions (kinematic and dynamic similitude)

I . GEOMETRIC SIMILARITY (SIMILITUTDE) The model and its prototype be identical in **shape** but differ only in **size.** Geometric similitude exists between model and prototype if the ratios of all corresponding dimensions in model and prototype are equal.

- $L_r = Scale\,\text{ratio} = \frac{L_p}{L_m}$, $L_r^{\,2} = \frac{A_p}{A_m} = \frac{L_p^{\,2}}{L_m^{\,2}}$

    - Note that $L_r = \frac{L_p}{L_m}$ scale ratio ,

  - the reciprocal of this $\lambda = \frac{L_m}{L_p}$ will be referred to as the **model ratio or model scale**.

II. **KINEMATIC SIMILARITY:** Kinematic similarity implies geometric similarity and in addition it implies that the ratio of the **<u>velocities</u>** at all corresponding points in the flow is the same.

Velocity ratio $= V_r = \frac{V_p}{V_m} = \frac{L_r}{T_r}$ and its value in terms of L_r will be determined by dynamic considerations. T is dimensionally L/V .

The time scale is $T_r = \frac{L_r}{V_r}$

Acceleration scale is $a_r = \frac{L_r}{T_r^{\,2}}$

Discharge $= \frac{Q_p}{Q_m} = Q_r = \frac{L_r^{\,3}}{T_r}$

III DYNAMIC SIMILARITY: If two systems are dynamically similar, corresponding FORCES must be in the same ratio in the two. Forces that may act on a fluid element are:

(i) Gravity $F_G = mg = \rho L^3 g$

(ii) Pressure $F_P = (\Delta p) L^2$

(iii) Viscosity $F_V = \mu\left(\frac{du}{dy}\right)A = \mu\left(\frac{V}{L}\right)L^2 = \mu VL$

(iv) Elasticity $F_E = E_V A = E_V L^2$

(v) Surface Tension $F_T = \sigma L$

(vi) Inertia $FI = ma = \rho L^3 \frac{L}{T^2} = \rho L^4 T^{-2} = \rho V^2 L^2$

The conditions required for complete similitude are developed from Newton's second law of motion $\sum F_x = ma_x$

REYNOLDS NUMBER

Considering the ratio of Inertia forces to viscous forces the parameter obtained is called Reynolds Number R_e or N_R. The ratio of these two forces is:

R_e or $N_R = \frac{F_I}{F_V} = \frac{InertiaForces}{ViscousForces} = \frac{L^2V^2\rho}{LV\mu} = \frac{LV\rho}{\mu} = \frac{LV}{\upsilon}$ is a dimensionless number.

EXAMPLE

1. If the Reynolds number of a model and its prototype are the same find an expression for V_r, T_r, and a_r.
2. Let us consider the drag force F_D exerted on a sphere as it moves through a viscous liquid

The relationship of variables is our concern our approach is to ***satisfy dimensional homogeneity, i.e. Dimensions on LHS=Dimensions on the RHS***

TWO METHODS ARE AVAILABLE

Let us consider the drag force F_D exerted on a sphere as it moves through a viscous liquid

1. **RAYLEIGH METHOD**

2. **BUCKINGHAM π THEOREM**

<u>SOLUTION TO PROBLEM 2</u>

(i) Visualize the physical problem

(ii) Consider what physical factors influence the drag force

(a) the size of the sphere

(b) the velocity of the sphere

(c) fluid properties , density and viscosity

<u>RAYLEIGH METHOD</u>

$F_D = f(D, V, \rho, \mu)$ could be written as power equation

$F_D = CD^a V^b \rho^c \mu^d$ where C is a dimensionless constant. Using MLT system and substituting the proper dimensions.

$F = ma$

$$\frac{ML}{T^2} = L^a \left(\frac{L}{T}\right)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{LT}\right)^d$$

To satisfy dimensional homogeneity the exponents of each dimension must be identical on both sides of the equation.

M:	$1 = c + d$
L:	$1 = a + b - 3c - d$
T:	$-2 = -b - d$

We have 3 equations and 4 unknowns

Express three of the unknowns in terms of the fourth.

Solving for a, b, c, in terms of d

$$a = 2 - d$$
$$b = 2 - d$$
$$c = 1 - d$$
$$F_D = CD^{2-d}V^{2-d}\rho^{1-d}\mu^{d}$$

Grouping variables according to their exponents

$$F_D = C\rho D^2 V^2 \left(\frac{VD\rho}{\mu}\right)^d$$

The quantity $\dfrac{VD\rho}{\mu} = R_e$

The power equation can be expressed as $F_D = f(R_e)\rho D^2 V^2$ or $\underline{\left(\dfrac{F_D}{\rho D^2 V^2}\right) = f(R_e)}$

BUCKINHAM π-THEOREM

This theorem states that if there are n dimensional variables I a dimensionally homogeneous equation, described by m fundamental dimensions, they may be grouped in n-m (n minus m) dimensionless groups.

$$
\begin{aligned}
&F_D = f(D,V,\rho,\mu)\\
&n = 5 = F_D, D, V, \rho, \mu\\
&m = 3 = M, L, T\\
&n - m = 2
\end{aligned}
$$

Buckingham referred to these dimensionless groups as πterms. The advantage of the π theorem is that it tells one ahead of time how many dimensionless groups are to be expected.

$$
\begin{aligned}
&f^{'}(F_D, D, V, \rho, \mu) = 0\\
&n = 5\\
&m = 3\\
&n - m = 2\\
&\phi(\pi_1, \pi_2) = 0
\end{aligned}
$$

Arrange the five parameters into 2 dimensionless groups, ***taking*** ρ ***, D and V as the primary variables.***

***It is generally advantageous to choose primary variables that relate to Geometry, Kinematics and Mass.

$$
\begin{aligned}
&\pi_1 = \rho^{a1} D^{b1} V^{c1} \mu^{d1}\\
&\pi_2 = \rho^{a2} D^{b2} V^{c2} F_D^{\;d2}
\end{aligned}
$$

Since the π 's (pi's) are dimensionless they can be replaced with $M^0 L^0 T^0$

Working with π_1

$$M^0L^0T^0 = \left(\frac{M}{L^3}\right)^{a1} L^{b1}\left(\frac{L}{T}\right)^{c1}\left(\frac{M}{LT}\right)^{d1}$$
$$M : 0 = a1 + d1$$
$$L : 0 = -3a1 + b1 + c1 - d1$$
$$T : 0 = -c1 - d1$$

Solving for a1, b1 and c1 in terms of d1

$$a1 = -d1$$
$$b1 = -d1$$
$$c1 = -d1$$
$$\pi_1 = \rho^{-d1}D^{-d1}V^{-d1}\mu^{d1} = \left(\frac{\mu}{\rho DV}\right)^{d1} = \left(\frac{\rho DV}{\mu}\right)^{d1} = N_R = R_e = \mathrm{Re}\,ynoldsNumber$$

Working in a similar fashion with π_2

$$\pi_2 = \frac{F_D}{\rho D^2V^2}$$
$$\frac{F_d}{\rho D^2V^2} = \phi''(N_R)$$
$$F_D = \phi''(N_R)\rho D^2V^2$$

Take Note

(a) That dimensional analysis does not provide a complete solution to fluid problems

(b) It provides a partial solution only

(c) That the success of dimensional analysis depends entirely on the ability of the individual using it to define the parameters that are applicable.