UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA

DEPARTMENT OF MATHEMATICS

Course Code	**MTS 101**
Course Title	**ALGEBRA**
Number of Units	**3 units**
Course Duration	**3 Hours per week for 2 Weeks**
Course Lecturer	**Professor J. A. Oguntuase**
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Course Outline:

Real Numbers: Numbers systems from Natural to reals. Operations on real numbers. Indices, logarithms and surds.

Prerequsite: Good WAEC/NECO result in Mathematics

Textbooks

1. Robert A. Adams; Calculus, 4th Edition, Addison-Wesley Longman Ltd., Ontario, Canada

What is expected of the Student:

Students are expected to attend all lectures and complete all assignments and examinations. No aids are permitted in examinations.

Evaluation of Student Performance:

1. Midsemester Examination: 20% (Date and lenght to be determined).
2. Written Assignments: 10% (Dates to be announced).
3. Final Examination: 70% (date and time to be determined and fixed by TIMTEC).

1 Real Numbers and the Real Line

Calculus depends on the properties of the real number system. **Real numbers** are numbers that can be express as decimals, for example

$$\begin{aligned} 5 &= 5.00000... \\ -\frac{3}{4} &= -0.750000... \\ \frac{1}{3} &= 0.3333... \\ \sqrt{2} &= 1.4141... \\ \pi &= 3.14159... \end{aligned} \tag{1}$$

In (1) each of the three dots indicates that the sequence of decimal digits goes on forever. Observe that the patterns of the first three numbers in (1) above are obvious since we know that the subsequent digits are. However for $\sqrt{2}$ and π there are no obvious pattern. Geometrically, we can represent the real numbers as points on the real line (as shown below).

The symbol $\mathbb{R}$ is used to denote either the real number system or equivalently, the real line.

Properties of the Real Numbers

Basically the properties of the real number system fall into three categories namely: algebraic, order and completeness properties.

Algebraic properties. You are familiar with the algebraic properties which assert that real numbers can be added, subtracted, multiplied, and divided (except by zero) to produce more real numbers and the usual rules of arithmetic are valid.

Order properties. This refer to the order in which the numbers appear on the real line. For example, if x lie to the left of y, then we say that "x is less than y" or "y is greater than x". These statements are written symbolically as $x < y$ and $y > x$, respectively. Thus the inequality $x \leq y$ would mean that either $x < y$ or $x = y$.

The order properties of the real numbers are summarized in the following rules for inequalities.

Rules for Inequalities

If a, b and c are real numbers, then

$$\begin{aligned}
1.\ a &< b \Longrightarrow a+c<b+c \\
2.\ a &< b \Longrightarrow a-c<b-c \\
3.\ a &< b \text{ and } c>0 \Longrightarrow ac<bc \\
4.\ a &< b \text{ and } c<0 \Longrightarrow ac>bc, \text{ in particular, } -a>-b \\
5.\ a &> 0 \Longrightarrow \frac{1}{a}>0 \\
6.\ 0 &< a<b \Longrightarrow \frac{1}{b}<\frac{1}{a}
\end{aligned}$$

Rules $1-4$ and 6 (for $a > 0$) also hold if $<$ and $>$ are replaced by $\leq$ and $\geq$.

Remark 1 *Observe that the rule for multiplying (or dividing) an inequality by a number. If the number is positive the inequality is preserved, if the number is negative, the inequality is reversed.*

Completeness properties of the Real Numbers

This is more subtle or difficult to understand. If A is any set of real numbers having at least one number in it and if there exists a real number y with the property that $x \leq y$ for every x in A, then there exists a smaller number y with the same property. In other words, this says that there is no holes or gaps on the real line- every point corresponds to a real number.

The set of real numbers has some important subsets as follows:

1. The natural numbers $\mathbb{N}$ or positive integers $\mathbb{Z}^+$, namely, the numbers, $1, 2, 3, 4, ...$

2. The integers $\mathbb{Z}$, namely, the numbers $0, \pm1, \pm2, \pm3, ...$

3. The rational numbers $\mathbb{Q}$, that is, numbers that can be expressed in the form of a fraction $\frac{m}{n}$, where m and n are integers, and $n \neq 0$. Precisely, the rational numbers are those real numbers with decimal expressions that are either:

(i) terminating (that is, ending with an infinite string of zeros), for example $\frac{3}{4} = 0.750000..$ or

(ii) repeating (that is, ending with a string of digits that repeat over and over), for example, $\frac{23}{11} = 2.090909... = 2.\overline{09}$. (The bar indicates the pattern of the repeating digits).

Real numbers that are not rational are called irrational numbers.

Example 2 *Show that each of the following numbers* (a) $1.323232... = 1.\overline{32}$ *and* (b) $0.3405405405... = 0.3\overline{405}$ *is a rational number by expressing it as a quotient of two integers.*

Solution 3 *(a) Let* $x = 1.323232...$ *Then* $x - 1 = 0.323232...$ *and*

$$100x = 132.323232... = 132 + 0.323232... = 132 + x - 1.$$

Therefore,

$$99x = 131 \text{ and } x = \frac{131}{99}.$$

(b) Let $y = 0.3405405405...$ *Then* $10y = 3.405405405...$ *and* $10y - 3 = 0.405405405...$

Also

$$10000y = 3405.405405405... \ = 3405 + 10y - 3.$$

Therefore,

$$9990y = 3405 \quad \text{and} \quad y = \frac{3405}{9990} = \frac{63}{185}.$$

Remark 4 *Note that the set of rational numbers has all the algebraic and order poroperties of the real numbers but does not have the completeness property. This is because, there is no rational number whose square is* 2. *Hence, there is a "hole" on the "rational numbers" since* $\sqrt{2}$ *exists as irrational number (prove this).*

Because the real numbers has no such "holes", it is the appropriate setting for studying limits and therefore calculus.

Intervals

A subset of the real line is called an interval if it contains at least two real numbers and also contains all real numbers between any two of its elements.

If a and b are real numbers such that $a < b$, then we define the followings:

(i) the **open interval** from a to$b$, denoted by (a, b) consists of all real numbers x satisfying $a < x < b$.

(ii) the **closed interval** from a to$b$, denoted by $[a, b]$ consists of all real numbers x satisfying $a \leq x \leq b$.

(iii) the **half-open interval** $[a, b)$ consists of all real numbers x satisfying $a \leq x < b$.

(iv) the **half-open interval** $(a, b]$ consists of all real numbers x satisfying $a < x \leq b$.

The end points of an interval are called the boundary points. Observe that the intervals $(i) - (iv)$ above are finite intervals since each of them has finite lenght $b - a$. Note that an interval may be infinite, e.g. (a, ∞), $\mathbb{R} = (-\infty, \infty)$. The symbol ∞ ("infinity") does not denote a real number and so we do not use a square bracket at an infinite end of the interval. Draw the graph of the above (exercise).

Example 5 *Solve the following inequalities. Express the solution sets in terms of intervals and graph them.*
(*a*) $2x-1>x+3$ (*b*) $-\frac{x}{3}\geq 3x-1$ (*c*) $\frac{2}{x-1}\geq 5$.

Solution 6

$$
\begin{array}{rrcll}
(a) & 2x-1 & > & x+3 & \textit{Add 1 to both sides} \\
 & 2x & > & x+4 & \textit{Subtract } x \textit{ from both sides} \\
 & x & > & 4 & \textit{The solution set is the interval } (4,\infty).
\end{array}
$$

$$
\begin{array}{rrcll}
(b) & -\dfrac{x}{3} & \geq & 2x-1 & \textit{Multiply both sides by } -3. \\
 & x & \leq & -6x+3 & \textit{Add } 6x \textit{ to both sides.} \\
 & 7x & \leq & 3 & \textit{Divide both sides by } 7. \\
 & x & \leq & \dfrac{3}{7} & \textit{The solution set is the interval } (-\infty,\dfrac{3}{7}).
\end{array}
$$

(*c*) *Multiply both sides of* $\frac{2}{x-1}\geq 5$ *by* $x-1$, *noting that the inequality we reverse if* $x-1<0$. *So, we break the problem into two cases as follows:*
CASE 1 $x-1>0$,*that is,* $x>1$.

$$
\begin{array}{rcll}
\dfrac{2}{x-1} & \geq & 5 & \textit{Multiply both sides by } x-1. \\
2 & \geq & 5x-5 & \textit{Add 5 to both sides.} \\
7 & \geq & 5x & \textit{Divide both sides by } 5. \\
\dfrac{7}{5} & \geq & x & \textit{The solutions for this case lie in the interval } (1,\dfrac{7}{5}].
\end{array}
$$

CASE 2 $x-1<0$,*that is,* $x<1$.

$$
\begin{array}{rcll}
\dfrac{2}{x-1} & \geq & 5 & \textit{Multiply both sides by } x-1. \\
2 & \leq & 5x-5 & \textit{Add 5 to both sides.} \\
7 & \leq & 5x & \textit{Divide both sides by } 5. \\
\dfrac{7}{5} & \leq & x & \textit{There are no numbers satisfying } x<1 \textit{ and } x\geq\dfrac{7}{5}.
\end{array}
$$

The solution set is the interval $(1,\frac{7}{5}]$.

Example 7 *Solve the system of inequalities:*
(a) $3\leq 2x+1\leq 5$, *and (b)* $3x-1<5x+3\leq 2x+15$.

Solution 8 *(a) We first solve the inequality* $3\leq 2x+1$ *to obtain* $2\leq 2x$, *so* $x\geq 1$. *Similarly, the inequality* $2x+1\leq 5$ *yields* $2x\leq 4$, *and so* $x\leq 2$. *The solution set of the system (a) is the closed interval* $[1,2]$.

(b) *We solve both inequalities as follows:*

$$\begin{aligned} 3x-1 &< 5x+3 \\ -1-3 &< 5x-3x \\ -4 &< 2x \\ -2 &< x \end{aligned} \qquad \textit{and} \qquad \begin{aligned} 5x+3 &\leq 2x+15 \\ 5x-2x &\leq 15-3 \\ 3x &\leq 12 \\ x &\leq 4 \end{aligned}$$

The solution set is the interval $(-2, 4]$.

Example 9 *Solve the inequality*

$$\frac{3}{x-1} < -\frac{2}{x}$$

and graph the solution set.

Solution 10 *This inequality can be written as*

$$\begin{aligned} \frac{3}{x-1} + \frac{2}{x} &< 0 \\ \frac{5x-2}{x(x-1)} &< 0. \end{aligned}$$

The solution will consist of those value of x for which the numerato and denominator have opposite signs. This resulted into two cases as follows:

CASE 1 $5x-2>0$ *and* $x(x-1)<0$. *In this cas we need* $x > \frac{2}{5}$ *and* $0<x<1$, *so the solution set is* $(\frac{2}{5}, 1)$.

CASE 2 $5x-2<0$ *and* $x(x-1)>0$. *In this case we need* $x<\frac{2}{5}$ *and either* $x<0$ *or* $x>1$, *so the solution set is* $(-\infty, 0)$.

The solution set of the given inequality is the union of these two intervals, namely, $(-\infty, 0) \cup (\frac{2}{5}, 1)$.

The Absolute Value

The **absolute value or magnitude** of a number x, denoted by $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0. \end{cases}$$

Geometrically, $|x|$ represents the (nonnegative) distance fro x to 0 on the real line.

Properties of Absolute Value

1. $|-a| = |a|$.
2. $|ab| = |a|\,|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

3. $|a \pm b| \leq |a| + |b|$. (the triangle inequality).

We now give examples of inequalities involving absolute values.

Example 11 *Solve (a)* $|2x+5| = 3$, *(b)* $|3x-2| \leq 1$.

Solution 12 *(a)* $|2x+5| = 3 \Longleftrightarrow 2x+5 = \pm 3$. *Hence, either* $2x = -3-5 = -8$ *or* $2x = 3-5 = -2$. *The solutions are* $x = -4$ *and* $x = -1$.

(b) $|3x-2| \leq 1 \Longleftrightarrow -1 \leq 3x-2 \leq 1$. *We then solve this pair of inequalities:*

$$\begin{array}{rcl} -1 & \leq & 3x-2 \\ -1+2 & \leq & 3x \\ \frac{1}{3} & \leq & x \end{array} \qquad \begin{array}{c} \\ \textit{and} \\ \\ \end{array} \qquad \begin{array}{c} 3x-2 \leq 1 \\ 3x \leq 1+2 \\ x \leq 1 \end{array}$$

Thus the solutions lie in the interval $[\frac{1}{3}, 1]$.

Example 13 *Solve the equation* $|x+1| = |x-3|$.

Solution 14 $|x+1| = |x-3|$ *implies that either* $x+1 = x-3$ *or* $x+1 = -(x-3)$. *The first of these equations has no solution while the second has the solution* $x = 1$.

Example 15 *What values of* x *satisfy the inequality* $\left|5 - \frac{2}{x}\right| < 3$?

Solution 16 *We have*

$$\begin{array}{rcll} \left|5 - \dfrac{2}{x}\right| & < & 3 \Longleftrightarrow -3 < 5 - \dfrac{2}{x} < 3 & \textit{Subtract 5 from each number.} \\ -8 & < & -\dfrac{2}{x} < -2 & \textit{Divide each number by } -2. \\ 4 & > & \dfrac{1}{x} > 1 & \textit{Take reciprocals.} \\ \dfrac{1}{4} & < & x < 1. & \end{array}$$

So the given inequality holds for all x *in the open interval* $(\frac{1}{4}, 1)$.

Exercise 17 *In Exercises 1-2, express the given rational number as a repeating decimal. Use a bar to indicate the repeating digits.*

1. $\frac{2}{9}$ *2.* $\frac{1}{11}$

In Exercises 3-4, express the given repeating decimal as a quotient of integers in the lowest terms.

3. $0.\overline{12}$ *4.* $3.\overline{27}$

In Exercises 5-10, solve the given inequality, given the solution set as an interval or union of intervals

5. $5x-3 \leq 7-3x$ *6.* $\frac{x+1}{x} \geq 2$ *7.* $\frac{1}{2-x} < 3$ *8.* $x^2 - x \leq 2$ *9.* $\frac{3}{x-1} < \frac{2}{x+1}$ *10.* $\frac{x}{2} \geq 1 + \frac{4}{x}$

In Exercises 11-14, solve the equations

11. $|x| = 3$ *12.* $|2t+5| = 4$ *13.* $\left|\frac{x}{2} - 1\right| = 1$ *14.* $|8-3s| = 9$

In Exercises 15-18, write the interval defined by the given inequality

15. $|x-1| \leq 2$ *16.* $|2x+5| < 1$ *17.* $\left|2 - \frac{x}{2}\right| < \frac{1}{2}$ *18.* $|\,3x-7| < 2$

19. Sove the equation $|x-1| = 1-x$

20. Show that the inequality $|a-b| \geq ||a|-|b||$ *holds for all real numbers a and b.*

INDICES

Consider the product of a number with itself, that is $a \times a$ is usually written as a^2. Similarly, $a \times a \times a = a^3$, i.e. the third power of a. Thus, the number which expresses the power is called the index.

Fundamental laws of indices

Let m and n be positive real numbers, then

(I) $a^m \times a^n = a^{m+n}$ e.g. $2^2 \times 2^3 = 2^{2+3} = 2^5$

(II) $a^m \div a^n = a^{m-n}$ e.g. $2^6 \div 2^3 = 2^{6-3} = 2^3$

(III) $(a^m)^n = a^{mn}$ e.g. $(2^2)^3 = 2^{2\times 3} = 2^6$

Other laws

(IV) $a^{\frac{1}{n}} = \sqrt[n]{a}$

(V) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(VI) $a^0 = 1$ for $a \neq 0$

(VII) $a^{-n} = \frac{1}{a^n}$

As application of the above laws we have the following examples.

Example 18 *Simplify*

1. $2^0 \times 3^3$

2. $2^{\frac{1}{2}} \times 8^{\frac{1}{2}}$

3. $64^{\frac{1}{2}} \times 4^{-\frac{1}{2}}$

Solution 19 *1.* $2^0 \times 3^3 = 1 \times 27 = 27$

2. $2^{\frac{1}{2}} \times 8^{\frac{1}{2}} = (2\times 8)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4$

3. $64^{\frac{1}{2}} \times 4^{-\frac{1}{2}} = \frac{64^{\frac{1}{2}}}{4^{\frac{1}{2}}} = \left(\frac{64}{4}\right)^{\frac{1}{2}} = 16^{\frac{1}{2}} = 4$

Example 20 *Solve the equations*

1. $16 = 8^n$

2. $9^x - 4.3^x + 3 = 0.$

Solution 21 *1.* $16 = 8^n$ *can be written as*
$2^4 = (2^3)^n \Longrightarrow 2^4 = 2^{3n}$
It follows that $4 = 3n$, *and so* $n = \frac{4}{3}$.

Solution 22 *2.* $9^x - 4.3^x + 3 = 0$ *can be written as*
$(3^2)^x - 4.3^x + 3 = 0.$
$\Longrightarrow \quad (3^x)^2 - 4.3^x + 3 = 0 \qquad (*)$
Let $3^x = y$, *then* $(*)$ *becomes*
$y^2 - 4y + 3 = 0$
$\Longrightarrow (y-1)(y-3) = 0.$
Hence, $y = 1$ *and* $y = 3$
Case1
$y = 1 \Longrightarrow 3^x = 1$
$\Longrightarrow 3^x = 3^0$
$\Longrightarrow x = 0.$
Case 2

Solution 23 $y = 3 \Longrightarrow 3^x = 3$
$\Longrightarrow 3^x = 3^1$
$\Longrightarrow x = 1.$

So, the solution of the equation is $x = 0$ or $x = 1$.

Example 24 *Evaluate*

$$\frac{4.2^{n+1} - 2^{n+2}}{2^{n+2} - 2^n}$$

Solution 25

$$\frac{4.2^{n+1} - 2^{n+2}}{2^{n+2} - 2^n} = \frac{2^2.2^n.2^1 - 2^n.2^2}{2^n.2^2 - 2^n} = \frac{2^n(2^3 - .2^2)}{2^n(2^2 - 1)} = \frac{(2^3 - .2^2)}{(2^2 - 1)} = \frac{8-4}{4-1} = \frac{4}{3}.$$

Exercise 26 1. *Evaluate* *(a)* $2^{\frac{1}{3}} \times 4^{\frac{1}{3}}$ *(b)* $27^{\frac{1}{2}} \times 3^{-\frac{1}{2}}$.

2. *Solve the equations* *(a)* $4^n = 8$ *(b)* $4^x = 5.2^x + 2^2 = 0$ *(c)* $9^x - 4.3^{x+1} + 3^3 = 0.$

3. *Simplify* $\left(x^4y^2z^{-3}\right)^{\frac{1}{2}}\left(\sqrt{x^5y^2z}\right) \div xz^{\frac{7}{2}}$

SURDS

A number which cannot be expressed as a fraction is called an irrational number in the form of roots are called surds. For example $\sqrt{2}, \sqrt{7}$, e.t.c.

Operation with surds

1. $\sqrt[m]{ab} = \sqrt[m]{a} \times \sqrt[m]{b}$

2. $\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}$

3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Note that the n^{th} root of a number "a"written as $\sqrt[n]{a}$ is a number which when multiplied by itself n times gives a.

A surd can be expresses in terms of another simpler surds as shown below:

1. $\sqrt{8} = \sqrt{2 \times 2 \times 2} = \sqrt{2^2 \times 2} = 2\sqrt{2}$

2. $\sqrt[3]{32} = \sqrt[3]{2 \times 2 \times 2 \times 2 \times 2} = \sqrt[3]{2^3 \times 4} = \sqrt[2]{4}$

FACTS: $\sqrt[n]{a^n} = a^{\frac{n}{n}} = a^1 = a^1 = a.$

Exercise 27 *Simplyfy (a)* $\sqrt{32}$ *(b)* $\sqrt{500}$ *(c)* $\sqrt[2]{128}$

Rationalization

Rationalization is the process of removing surd from the denominator of an expression. This is done by multiplying the numerator and denominator of the expression by the conjugate of the surd in the denominator. To illustrate this process, we give the following examples:

Example 28 *Rationalize (a)* $\frac{1}{\sqrt{2}}$ *(b)* $\frac{2}{5\sqrt{2}}$

Solution 29 *(a)* $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
(b) $\frac{2}{5\sqrt{2}} = \frac{2}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{5\times 2} = \frac{\sqrt{2}}{5}.$

Note:
The conjugate of $\sqrt{2}$ is $\sqrt{2}$
The conjugate of $5\sqrt{3}$ is $\sqrt{3}$
The conjugate of $2\sqrt{3} + \sqrt{5}$ is $2\sqrt{3} - \sqrt{5}$
The conjugate of $\sqrt{7} - \sqrt{5}$ is $\sqrt{7} + \sqrt{5}$.

Example 30 *Rationalize* $\frac{3}{\sqrt{3}+5\sqrt{2}}$.

Solution 31

$$\begin{aligned}
\frac{3}{\sqrt{3}+5\sqrt{2}} &= \frac{3}{\sqrt{3}+5\sqrt{2}} \times \frac{\sqrt{3}-5\sqrt{2}}{\sqrt{3}-5\sqrt{2}} = \frac{3\left(\sqrt{3}-5\sqrt{2}\right)}{3-5\sqrt{6}+5\sqrt{6}-50} \\
&= \frac{3\left(\sqrt{3}-5\sqrt{2}\right)}{3-50} = \frac{3\left(\sqrt{3}-5\sqrt{2}\right)}{-47} = \frac{-3\left(\sqrt{3}-5\sqrt{2}\right)}{47} \\
&= \frac{-3\sqrt{3}+15\sqrt{2}}{47}.
\end{aligned}$$

Equation involving surds
We illustrate with example.

Example 32 *Solve for* x *if* $\sqrt{x}+\sqrt{x+7}=7$

Solution 33 *By squaring both sides we have*

$$\begin{aligned}
\left(\sqrt{x}+\sqrt{x+7}\right)^2 &= 7^2\\
x+x+7+2\sqrt{2(x+7)} &= 49\\
2x-42 &= -2\sqrt{x(x+7)}\\
x-21 &= -\sqrt{x(x+7)}
\end{aligned}$$

Squaring both sides again gives

$$\begin{aligned}
(x-21)^2 &= x(x+7)\\
x^2-42x+441 &= x^2+7x\\
-49x+441 &= 0\\
49x &= 441\\
x &= \frac{441}{49}\\
x &= 9.
\end{aligned}$$

Example 34 *Find the square root of* $5-2\sqrt{6}$.

Solution 35 *Let the square root of* $5-2\sqrt{6}$ *be* $\sqrt{x}-\sqrt{y}$. *That is*

$$\sqrt{\sqrt{5-2\sqrt{6}}}=\left(\sqrt{x}-\sqrt{y}\right).$$

Squaring both sides to have

$$5-2\sqrt{6}=x+y-2\sqrt{xy}$$

Equating the rational and the irrational parts to have

$$\begin{aligned}
x+y &= 5 \qquad (2)\\
\sqrt{xy} &= \sqrt{6} \qquad (3)
\end{aligned}$$

Squaring both sides of equation (3) we obtain

$$\begin{aligned}
x+y &= 5 \qquad (4)\\
xy &= 6 \qquad (5)
\end{aligned}$$

From (4), $y=5-x$. *Substitute this value into (5) we have*

$$\begin{aligned} 6 &= x(5-x) \\ x^2-5x+6 &= 0 \\ (x-2)(x-3) &= 0 \\ x &= 2 \textit{ and } x=3 \end{aligned}$$

Substitute the values of x *into (4) to have* $y=3$ *when* $x=2$ *and* $y=2$ *when* $x=3$.

The root of $\sqrt{5-2\sqrt{6}} = \pm\left(\sqrt{2}+\sqrt{3}\right)$.

Exercise 36 *1. Express without roots in the denominator (a)* $\frac{1}{\sqrt{3}}$ *(b)* $\frac{4}{5\sqrt{2}}$ *(c)* $\frac{1}{2-\sqrt{3}}$ *(d)* $\frac{4}{2\sqrt{3}+\sqrt{2}}$.

2. *Solve the equations: (a)* $\sqrt{x+5} = 5-\sqrt{x}$ *(b)* $\sqrt{2x+3}-\sqrt{x-2} = 2$.

3. *Find the square roots of (a)* $8+2\sqrt{5}$ *(b)* $18-12\sqrt{2}$.

4. *Find the square root of* $11-2\sqrt{30}$ *and hence show that*

$$\frac{1}{\sqrt{11-2\sqrt{30}}} = \frac{3}{\sqrt{5}-\sqrt{2}} + \frac{2\sqrt{2}}{1+\sqrt{3}}.$$

Logarithms

We already know that $100 = 10^2$, $1000 = 10^3$, *e.t.c.* In general, if $y = 10^x$, then x is called the logarithms of y to base 10. Thus, $100 = 10^2 \Longrightarrow \log 100 = 2$.

Similarly, $1000 = 10^3 \Longrightarrow \log 1000 = 3$.

Observe that the logarithm of a number need not be to base 10; in fact, it can be to any base. Hence, the logarithms of a number to base b is the power to which b must be raised to give the number. Thus, the logarithms of a number to base 10 is the power to which 10 must be raised to give the number.

Formulae connecting logarithms

1. $\log_a p + \log_a q = \log_a pq$
2. $\log_a p - \log_a q = \log_a \frac{p}{q}$
3. $\log_a p^m = m \log_a p$
4. $\log_p q = \frac{1}{\log_q p}$
5. $\log_p q = \frac{\log_a q}{\log_a p}$, where a is any positive integer. This is called the change of base formula

The proofs of the above are trivial and so left as exercise.

Note: $\log_a a^m = m$ and $\log_a a = 1$.

As illustration of the above formulae, we consider the following examples.

Example 37 *Evaluate* *(a)* $\log_3 27$ *(b)* $\log_{\frac{1}{3}} 9$ *(c)* $\log_{y^2} y$.

Solution 38 *(a)* $\log_3 27 = \log_3 3^3 = 3\log_3 3 = 3.$
(b) $\log_{\frac{1}{3}} 9 = \log_{\frac{1}{3}} 3^2 = \log_{\frac{1}{3}} \left(\frac{1}{3}\right)^{-2} = -2\log_{\frac{1}{3}} \frac{1}{3} = -2.$
(c) $\log_{y^2} y = \log_{y^2} \left(y^2\right)^{\frac{1}{2}} = \frac{1}{2}\log_{y^2} y^2 = \frac{1}{2}.$

Example 39 *Evaluate* *(a)* $\log_6 12$ *(b)* $\log_3 24$ *(c)* $\log_7 35$.

Solution 40 *(a)* $\log_6 12 = \frac{\log_{10} 12}{\log_{10} 6} = \frac{1.0792}{0.7782} = 1.387$
(b) $\log_3 24 = \frac{\log_{10} 24}{\log_{10} 3} = \frac{1.3802}{0.4771} = 2.893$
(c) $\log_7 35 = \frac{\log_{10} 35}{\log_{10} 7} = \frac{1.5441}{0.8451} = 1.827 \approxeq 1.83$

Example 41 *Solve the equation*

1. $3^x = 7.83$
2. $\log_3 x + \log_x 3 = 2.5$
3. $4^x - 5.2^x + 4 = 0.$

Solution 42 *1.* $3^x = 7.83$
Take the log on both sides to base 10,*we obtain*

$$\begin{aligned}
\log_{10} 3^x &= \log_{10} 7.83 \\
x\log_{10} 3 &= \log_{10} 7.83 \\
x &= \frac{\log_{10} 7.83}{\log_{10} 3} \\
&= \frac{0.8938}{0.4771} \\
&= 1.873 \approxeq 1.87
\end{aligned}$$

2. $\log_3 x + \log_x 3 = 2.5$
Let $p = \log_3 x$, *then we have*

$$\begin{aligned}
\frac{1}{p} + p &= \frac{5}{2} \\
&\implies \frac{p^2+1}{p} = \frac{5}{2} \\
2p^2 - 5p + 2 &= 0 \\
(2p-1)(p-2) &= 0 \\
p &= \frac{1}{2} \ \mathit{or} \ \ 2
\end{aligned}$$

But $p = \log_3 x \Longrightarrow x = 3^p$.
Case 1: When $p = 2$, *then* $x = 3^2 = 9$.
Case 2: When $p = \frac{1}{2}$, *then* $x = 3^{\frac{1}{2}} = \sqrt{3}$.
Hence the solution is $x = 9$ *or* $\sqrt{3}$.
3.

$$\begin{aligned} 4^x - 5.2^x + 4 &= 0 \\ (2^2)^x - 5.2^x + 4 &= 0 \\ (2^x)^2 - 5.2^x + 4 &= 0 \end{aligned}$$

Put $y = 2^x$, *then we have*

$$\begin{aligned} y^2 - 5y + 4 &= 0 \\ (y-4)(y-1) &= 0 \\ y &= 4 \ \ or \ \ 1 \end{aligned}$$

But $y = 2^x$, *so when* $y = 4$, *we obtain* $2^2 = 2^x \Longrightarrow x = 2$
Also when $y = 1$, *we obtain* $2^0 = 2^x \Longrightarrow x = 0$.
The solution is $x = 2$ *or* 0.

Example 43 *Show that*

$$2\log_{10} 6 + 3\log_{10} 2 = \log_{10} 288$$

hence show that

$$\log_{10} 17 \approxeq \frac{1}{2}\left(\log_{10} 6 + 3\log_{10} 2\right).$$

Solution 44 *Observe that*

$$\begin{aligned} 17^2 &= 289 \approxeq 288. \\ \log_{10} 17^2 &\approxeq \log_{10} 288 = 2\log_{10} 6 + 3\log_{10} 2 \\ 2\log_{10} 17 &\approxeq 2\log_{10} 6 + 3\log_{10} 2 \end{aligned}$$

Divide through by 2, we obtain

$$\log_{10} 17 \approxeq \frac{1}{2}\left(\log_{10} 6 + 3\log_{10} 2\right).$$

which is required.

MTS 101 LECTURE 4 : MATRICES AND MATRIX ALGEBRA

DEFINITION

A matrix is a rectabgular array of the elements of a field (i.e. on array of numbers). Thus if m, n are two positive integers ≥ 1 and F is a field ($\mathbb{R}$ *or* $\mathbb{C}$) then the array:

$$= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & --- & - \\ - & - & --- & - \\ a_{m1} & a_{m2} & --- & a_{mn} \end{pmatrix}$$

is called an m x n matrix in F (since it contains m rows and n columns)

Its first row is $(a_{11} \quad a_{12} \quad \dots \quad a_{1n})$ and first column is $\begin{pmatrix} a_{11} \\ a_{12} \\ - \\ - \\ a_{m1} \end{pmatrix}$

The numbers that constitute the matrix are called its ELEMENTS.

Let a_{ij} denote the element of the matrix in the i^{th} row an j^{th} column. Then for ease of notation we can denote our m x n matrix by

(a_{ij}) i = 1,2,….,n j=1,2,…..,m or simply by capital letter $A_{m \times n}$

Order

The order of a matrix is the no of rows and columns e.g (a_{ij}) is of order m x n.

If m=n, then the matrix is called a SQUARE MATRIX of order n.

Definition 2: Row and Column Matrices

A rectangle matrix consisting of only a single row is called a ROW MATRIX e.g. (1,2,3,4). Similarly, a rectangle matrix consisting of a single column is called a COLUMN MATRIX e.g.

$$\begin{pmatrix} 3 \\ 5 \\ 7 \\ 4 \end{pmatrix}$$

Definition 3: Null Matrix

This is a matrix having each of its elements = 0 e.g $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Definition 4: Diagonal Element, Diagonal Matrix

The elements a_{ij} of a matrix (a_{ij}) are called its DIAGONAL ELEMENTS (or elements of the main diagonal)

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **, a_{11} = 1, a_{22} =5, a_{33} = 9**

A square matrix in which all the elements other than the diagonal elements are zero is called a DIAGONAL MATRIX.

$$\text{Viz} \quad = \begin{pmatrix} d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & d_3 & 0 & --- & 0 \\ - & - & --- & - & --- & 0 \\ 0 & 0 & 0 & 0 & --- & d_n \end{pmatrix} \quad \text{e.g.} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Such a matrix is denoted $(d_1, d_2, \dots, d_n)$ or (d_i, d_{ik}) for I, k = 1,2, …n where $d_{ii}=1$, $d_{ik} = 0$ $(i \neq k)$

NB: Its diagonal elements may also be zero

e.g. i. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ii. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ iii. $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

are diagonal matrices.

Definition 5: Scalar and Scalar Matrix

A diagonal matrix where diagonal elements are all equal is called a SCALAR MATRIX.

e.g. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Definition 6: Identity Matrix (or Unit Matrix)

A diagonal matrix whose elements are each equal to 1 is called and IDENTITY MATRIX. It is denoted In (or I_{nxn}) .

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Definition 7: Symmetry

A square matrix where elements are arranged symmetrically about the main diagonal is called a SYMMETRIC MATRIX.

e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

On the other hand, if for a square matrix, there is no symmetric about the main diagonal but for every element a_{ij} on one side of the main diagonal, there is a corresponding – a_{ij} on the other side, then the matrix is a SKEW-SYMMETRIC MATRIX.

Furthermore, the diagonal elements are all zero

e.g.
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & -\frac{1}{2} & 2 \\ -2 & \frac{1}{2} & 0 & 1 \\ -3 & 2 & -1 & 0 \end{pmatrix} \rightarrow \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

Definition 8: Triangular matrix

A square matrix whose elements a_{ij} are all zero whenever $i < j$ is called a LOWER TRIANGULAR MATRIX.

A square matrix whose elements $a_{ij} = 0$ whenever $i > j$ is called an UPPER TRIANGULAR MATRIX.

Hence, a diagonal matrix is both upper and lower matrix.

e.g. $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 3 & 2 \end{bmatrix}$, $\rightarrow$ Lower Triangular Matrix

$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 3/4 \end{bmatrix}$, $\begin{bmatrix} 0 & 1/2 \\ 0 & 3 \end{bmatrix}$, $\rightarrow$ Upper Triangular Matrix

MATRIX ALGEBRA

Equality of Matrices

A and B are equal if

i. they are of the same order

ii. their corresponding elements are the same

Addition of Matrices

If A and B are of the same order, the their sum is a matrix C of the same order whose elements are the sums of the corresponding elements of A and B.

$\Longrightarrow C = A + B$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $\quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$C_{ik} = a_{ik} + b_{ik} \quad i = 1,2, \ldots\ldots\ldots m, \; k = 1,2, \ldots\ldots\ldots n$

* Only matrices of the same order can be added.

Properties of Matrix Addition

i. Matrix addition is commutative $\Longrightarrow$ $A + B = B + A$

ii. Matrix addition is Associative $\Longrightarrow$ $(A + B) + C = A + (B + C)$

iii. If 0 is a null matrix of the same order as A, the $A + 0 = 0 + A = A$

iv. To each A there exists a matrix B of the same order s.t $A + B = 0 = B + A$

(i) $\longrightarrow$ (iv) $\Longrightarrow$ Matrix addition is Abelian

Exercise 1.

Find the sum of these matrices and establish their Commutativity

i. $\begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & -3 \\ 7 & 5 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Exercise 2:

Establish the associativity of the following matrices

i. $\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 5 \\ 7 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 6 \\ 2 & 5 & 4 \end{bmatrix}$

ii. $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$

Exercise 3:

Establish the order of each matrix 1,2 and 4 and find the a_{11}, a_{12}, etc what are the diagonal elements.

Exercise 4:

Which of the following matrices are (i) Triangular Matrices, (ii) Unit Matrices (iii) null matrices and Scalar matrices.

i. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ iii. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 2 & 0 \end{bmatrix}$ iv. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

v. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ vi. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ vii. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ viii. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

ix. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 5:

i. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ Find a Matrix B such that A + B = 0

ii. If $A = \begin{bmatrix} 2 & 5 & 8 \\ 3 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -5 & -8 \\ -3 & -4 & -6 \end{bmatrix}$ Find A + B

Multiplication by Scalar

Let $A = (a_{ij})$ $i = 1,2,3,\ldots m$ be a matrix, $j = 1,2,3,\ldots n$

And Let α be a scalar (i.e. any number), then $\alpha A = C = (C_{ij})$

Where $\alpha a_{ij} = C_{ij}$ $i = 1,2, \ldots m$, $j = 1,2,\ldots n$

Example

If $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 4a_1 & 4a_2 & 4a_3 \\ 4b_1 & 4b_2 & 4b_3 \end{bmatrix}$

Properties

If A,B are matrices and α,β are scalars, then

i. $\alpha(A + \beta) = \alpha A + \alpha\beta$

ii. $(\alpha + \beta)A = \alpha A + \beta B$

iii. $(\alpha\beta) A = \alpha(\beta A)$ Examples to be given in class

DIFFERENCE OF TWO MATRICES

If two matrices A and B are of the same order, then the difference $A – B = A + (-B) = A + (-1)B$

Exercise 6:

i. For Exercice (2) above, find $(A – B) + C$

ii. $A – B – C$ iii. $2A + 3B$ iv. $A + 2B + {}^1/_2C$

Exercise 7:

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ and B =

i. Find a matrix C such that A + C is a diagonal matrix.

ii. Find a matrix D such that A + B = 2D.

iii. Find a Matrix E such that (A + B) + E is zero matrix.

MULTIPLICATION OF MATRICES

The product AB of two matrices exist if the number of columns of A = the number of rows of B.

e.g. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ x } \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix} = \begin{bmatrix} ap + bx & aq + by & ar + bz \\ cp + dx & cq + dy & cr + dz \end{bmatrix}$

Since A_{2x2} and B_{2x3} i.e, no. of column of A = no. of rows of B, then AB exist.

Let A = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ of order m x n and B = $\begin{bmatrix} b_{11} & \cdots & b_{iq} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nq} \end{bmatrix}$ of order n x q

Then AB is the matrix

$$C = \begin{bmatrix} C_{ii} & C_{iq} \\ C_{mi} & C_{mq} \end{bmatrix} \text{ of order } m \text{ x } q$$

in which the element C_{ij} is the sum of products (term by term) of elements of i^{th} row of A and the j^{th} column of B. Thus for the matrices A = (a_{ik}), B = (b_{kj}), the product AB is matrix C = (C_{ij})

where $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

* Multiplication is possible if no. of column of the first matrix = no. of rows of the second matrix.

Example:

If A = $\begin{bmatrix} 2 & 5 & 3 \\ 0 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -5 & 0 \\ 0 & 2 & 6 \end{bmatrix}$ Find (i). AB, (ii) BA (iii) A^2 (iv) $5B^2$

COURSE CODE: MTS 101

COURSE TITLE: Algebra

NUMBER OF UNITS: 3

COURSE DURATION: Three hours per week

COURSE DETAILS:

Course Coordinator: Dr.I. A. Osinuga ,Miss A. D. Akinola

Other Lecturer: Prof. J. A. Oguntuase,Dr. O. J. Adeniran, Dr. M. O. Omeike,Dr. S. A. Akinleye,Mr. E. Ilojide

TOPICS: Complex numbers: Algebra of Complex numbers,Argand diagram, De Moivre's theorem and nth root of complex numbers.

COURSE REQUIREMENTS:

This is a compulsory course for students in the department of Mathematics,Statistics and Computer science, Chemistry,Biochemistry,Food Science and Technology,Physics,WRM and all the departments in College of Engineering.

Complex Numbers

In order to solve equations such as

$$x^2 + 1 = 0$$

or

$$x^2 + 2x + 8 = 0$$

which have no root within the system of real numbers,the number system was extended further to the larger system of complex numbers.

By definition, a complex number is any number x that can be expressed in the form $x = a + ib$ where a and b are real and $i^2 = -1$.The symbol $\mathcal{C}$ is used to denote the system of complex numbers. a is referred to as the real part and b the imaginary part of $a + ib$. Note that the complex numbers include all real numbers. The real numbers can be regarded as complex numbers for which b is zero.

In $\mathcal{C}$, the solution of the equation

$$x^2 + 1 = 0$$

is then $x = \pm\sqrt{-1}$ i.e $x = \pm i$

Algebra of complex Numbers

Let $x = a + ib$ and $y = c + id$ be two complex numbers:

Equality of complex numbers: x and y are equal if their real and imaginary parts are equal i.e $a = c$ and $b = d$

Addition and subtraction of two complex numbers:

The sum of x and y is defined as a complex number $z = x+y = a+ib+c+id = a + c + i(b + d)$

Also,

$w = x - y = a + ib - (c + id) = a - c + i(b - d)$

Multiplication:

$x \times y = (a + ib) \times (c + id) = ac + i^2bd + ibc + iad$

$= ac - db + i(bc + ad)$

Division:

$\frac{x}{y} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c+id)(c-id)}$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Conjugate complex number:

$\overline{x} = a - ib$ is called the conjugate of x.

we have

$$x + \overline{x} = 2a$$

$$x - \overline{x} = 2ib$$

$$x\overline{x} = a^2 + b^2$$

Example: Express in the form $a + ib$

1. $(2 + 4i) + (5 - 2i) = 7 + 2i$

2. $(1 - 8i) - (7 + 2i) = (1 - 7) + (-8 - 2)i = -6 - 10i$

3. $\frac{2+3i}{3+2i} = \frac{2+3i}{3+2i} \times \frac{3-2i}{3-2i}$

$$= \frac{6 + 6}{9 + 4} + \frac{9 - 4}{9 + 4}i$$

$$\frac{12}{13} + \frac{5}{13}i$$

4. $(1+3i)^{-1} = \frac{1}{1+3i} = \frac{1}{1+3i} \times \frac{1-3i}{1-3i} = \frac{1}{10} - \frac{3}{10}i$

5. $(\frac{5(1+i)}{1+3i})^2 = (\frac{5+5i}{1+3i})(\frac{5+5i}{1+3i}) = 3 - 4i$

6. $\frac{2+3i}{i(4-5i} + \frac{2}{i} = \frac{2i-3+2(4i+5)}{-4+5i}$
   $= \frac{22}{41} - \frac{75}{41}i$

Note:

$$i^4 = i \times i^2 = -i$$

$$i^4 = 1$$

$$i^5 = i, i^6 = -1, i^7 = -i$$

and so on.

Example:

Find the solutions of the equation $4x^2 + 5x + 2 = 0$ in the form $\alpha + i\beta$.

Solution:

$x = \frac{-5 \pm \sqrt{-7}}{8}$

$$= -\frac{5}{8} + i\frac{\sqrt{7}}{8}$$

or

$$-\frac{5}{8} - i\frac{\sqrt{7}}{8}$$

Example:

Factorize $a^2 + 3b^2$ as a product of two complex numbers.

Solution:

$a^2 + 3b^2 = a^2 + (b\sqrt{3})^2$

$= (a + ib\sqrt{3})(a - ib\sqrt{3})$

The Argand Diagram

A complex number of the form $z = x+iy$ is specified by the two real numbers x and y.The complex number z may then be made to correspond to a point P with ordered pair of values (x, y) as the co-ordinates of the point P on the plane.

r is known as modulus of the complex number z and is written as $|z|$ or $modz$

$r = |z| = |x + iy| = \sqrt{x^2 + y^2}$

$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

The diagram which represents complex numbers is known as Argand diagram or Argand plane or complex plane.

The angle α between the line OP from the origin to the number and the x-axis is called the argument or amplitudes of the number z.

From the diagram,

$$x = rcos\alpha, y = rsin\alpha$$

$$x^2 + y^2 = r^2, \frac{y}{x} = tan\alpha$$

$$\alpha = argz = tan^{-1}\frac{y}{x}$$

Since on the circle, $\alpha + 2\Pi$ for any integer n,represent the same angle, it follows that the argument of a complex number is not unique such that $-\Pi < Arg(z) \leq \Pi$.

The complex number z can therefore be written as $z = x + iy = rcos\alpha + irsin\alpha = r(cos\alpha + isin\alpha), -\Pi < \alpha < \Pi$.

which is called the modulus-argument form or polar form or trigom=nometric form of the complex number z.

Let $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $Z_2 = r_2(cos\theta_2 + isin\theta_2)$ be two complex numbers. Then,

$z_1z_2 = r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)$

$$= r_1r_2[(cos\theta_1cos\theta_2 - sin\theta_1sin\theta_2) + i(sin\theta_1cos\theta_2 + cos\theta_1sin\theta_2)]$$

$$= r_1r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)] \qquad (*)$$

Therefore

$$|z_1z_2| = r_1r_2 = |z_1||z_2|$$

and

$$arg(z_1z_2) = \theta_1\theta_2 = argz_1 + argz_2$$

Thus when complex numbers are multiplied,their moduli are multiplied and their arguments are added.Also,

$$\frac{z_1}{z_2} = \frac{r_1(cos\theta_1 + isin\theta_1}{r_2(cos\theta_2 + isin\theta_2)}$$

$$\frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$$

$$|\frac{z_1}{z_2}| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$arg(\frac{z_1}{z_2}) = \theta_1 - \theta_2 = argz_1 - argz_2$$

Example: Find the moduli and the arguments of the following complex numbers.

1. $7i - 2$

   Solution:

   $|7i - 2| = \sqrt{7^2 + 2^2} = \sqrt{53} = 7.28$

   $arg(7i - 2) = tan^{-1}(\frac{7}{-2}) = 105.9^o$

2. $(7i - 2)(3 + 4i)$

   Solution:

   $|(7i - 2)(3 + 4i)| = |13i - 34|$

   $= \sqrt{34^2 + 13^2} = \sqrt{1325} = 36.40$

   $arg((7i - 2)(3 + 4i) = arg(13i - 34) = tan^{-1}(\frac{-13}{34}) = 159.1^o$

3. $\frac{7i-2}{3+4i}$

   Answer: $1.456, 52.8^o$

4. $(\frac{7i-2}{3+4i})^2$

   Answer: $2, 12, 105.6^o$

Example: Describe the locus of a complex number z which satisfies $|z-2| = 3|z+2i|$.

Solution:

Put $z = x + iy$. Then

$|z-2|^2 = 9|z+2i|^2$

$|(x-2)+iy|^2 = 9|x+i(y+2)|^2$

$(x-2)^2 + y^2 = 9[x^2 + (y+2)^2]$

$8x^2 + 8y^2 + 4x + 36y + 32 = 0$

$x^2 + y^2 + \frac{1}{2}x + \frac{9}{2}y + 4 = 0$

$(x+\frac{1}{4})^2 + (y+\frac{9}{4})^2 = 4 + (\frac{1}{4}^2) + (\frac{9}{4})^2 = \frac{18}{16}$

Locus is a circle,with center $(-\frac{1}{4}, -\frac{9}{4})$ and radius $\frac{3}{4}\sqrt{2}$

De Moivre's Theorem

In general, if there are n complex numbers $z_1, z_2, ..., z_n$ with moduli $r_1, r_2, ...r_n$ and arguments $\theta_1, \theta_2, ..., \theta_n$ respectively, repeated application of equation (*) yields

$$z_1.z_2...z_n = r_1...r_n[cos(\theta_1 + \theta_2 + ... + \theta_n) + isin(\theta_1 + \theta_2 + ... + \theta_n)]$$

In particular if

$$z_1 = z_2 = ... = z_n = z \qquad (say)$$

$$r_1 = r_2 = ... = r_n = r \qquad (say)$$

and

$$\theta_1 = \theta_2 = ... = \theta_n = \theta \qquad (say)$$

then we have

$$z^n = r^n(cosn\theta + isinn\theta)$$

i.e

$$z^n = [r(cos\theta + isin\theta)]^n$$

$$= r^n(cosn\theta + isinn\theta)$$

$$|z^n| = |z|^n, arg(z^n) = narg(z)$$

In particular,if $r = 1$ we get Demoivre's theorem

$$(cos\theta + isin\theta)^n = cosn\theta + isin\theta$$

for any positive integer n.

This result is also valid when n is any negative integer.Suppose n is a negative integer, say $-m$ where m is a positive integer.Then

$$(cos\theta + isin\theta)^{-m} = (\frac{1}{cos\theta + isin\theta})^m = \frac{1}{(cos\theta + isin\theta)^m}$$

$$(cosm\theta + isinm\theta)^{-1} = cosm\theta - isinm\theta$$

$$= cos(-m)\theta + isin(-m)\theta$$

which shows that Demoivre's theorem is valid when n is any negative integer.

Example: Express $cos3\theta$ and $sin3\theta$ in terms of powers of $cos\theta$ and $sin\theta$ respectively.

Solution:

By Demoivre's theorem we have

$$cos3\theta + isin3\theta = (cos\theta + isin\theta)^3$$

$$= cos^3\theta + 3cos^2\theta(isin\theta) + 3cos\theta(isin\theta)^2 + (isin\theta)^3$$

$$= cos^3\theta - 3sin^2\theta cos\theta + i(3cos^2\theta sin\theta - sin^3\theta)$$

The real part of this expression then gives

$$cos3\theta = cos^3\theta - 3cos\theta sin^2\theta$$

But

$$sin^2\theta = 1 - cos^2\theta$$

Therefore

$$cos3\theta = cos^3\theta - 3cos\theta(1 - cos^2\theta)$$

$$= cos^3\theta - 3cos\theta + 3cos^3\theta$$

$$4cos^3\theta - 3cos\theta$$

and the imaginary part gives

$$sin3\theta = 3cos62\theta sin\theta - sin^3\theta$$

$$= 3sin\theta(1 - sin^2\theta) - sin^3\theta$$

$$= 3sin\theta - 3sin^3\theta - sin^3\theta$$

$$= 3sin\theta - 4sin^3\theta$$

Example: Show that if $z = cos\theta + isin\theta$ and m is a positive integer then $z^m + \frac{1}{z^m} = 2cosm\theta$

Solution:

$z = cos\theta + isin\theta$

$$z^m = (cos\theta + isin\theta)^m = cosm\theta + isinm\theta \qquad (Demoivre'stheorem)$$

$$z^{-m} = cosm\theta - isinm\theta$$

$$z^m + z^{-m} = 2cosm\theta$$

Example:

$$\begin{aligned}(z + \frac{1}{z})^5 = z^5 + 5z^4.\frac{1}{z} + 10z^3.\frac{1}{z^2} + 10z^2.\frac{1}{z^3} + 5z.\frac{1}{z^4} + \frac{1}{z^5} \\ = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5} \\ = (z + \frac{1}{z^5}) + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z}) \\ = 2cos5\theta + 2 \times 5cos3\theta + 2 \times 10cos\theta\end{aligned}$$

$$= 2cos5\theta + 10cos3\theta + 20cos\theta$$

Example: Evaluate z^8 where $z = 1 + i\sqrt{3}$

Solution:

Writing z in the modulus-argument form we have $r = |z| = \sqrt{4} = 2$ and $argz = tan^{-1}\sqrt{3} = \frac{\Pi}{3}$

That is

$$z = 2(cos\frac{\Pi}{3} + isin\frac{\Pi}{3})$$

Therefore

$$z^8 = 2^8(cos\frac{\Pi}{3} + isin\frac{\Pi}{3})^8$$

By De Moivre's Theorem this becomes

$$z^8 = 2^8(cos\frac{8\Pi}{3} + isin\frac{8\Pi}{3})$$

$$= 256(-0.5 + 0.866i)$$

$$= -128 + 221.703i$$

Example: Factorize into linear factors $4z^2 + 4(1+i)z + 1 + 2i$

Solution:

$$4z^2 + 4(1+i)z + 1 + 2i = 4(z^2 + (1+i)z + \frac{1}{4}(1+2i))$$

First solve

$$z^2 + (1+i)z + \frac{1}{4}(1+2i) = 0$$

$$a = 1, b = 1+i, c = \frac{1}{4}(1+2i)$$

$$z = \frac{-1-i \pm \sqrt{(1+i)^2 - (1+2i)}}{2}$$

$$\frac{-1-i \pm \sqrt{-1}}{2} = \frac{1}{2}(-1, -i \pm i)$$

$$= -\frac{1}{2}$$

or

$$-\frac{1}{2} - i$$

$$\Longrightarrow \qquad 4z^2 + (4(1+i)z + 1 + 2i = 4(z + \frac{1}{2})(z + \frac{1}{2} + i)$$

Roots of Complex Numbers

Let $z^n = \alpha, n$ a positive integer and α a complex number (**)

Theorem: (Fundamental theorem of algebra)

Every polynomial of degree at least one with arbitrary numerical coefficients has at least one root which in the general sense is complex.

Consider (**), we have

$$z^n = \alpha = r(cos\theta + isin\theta)$$

so that

$$z = r_o(cos\theta_o + isin\theta_o) \qquad provided \qquad \alpha \neq 0$$

Then by De Moivre's theorem

$$r_o^n(cosn\theta_o + isinn\theta_o) = r(cos\theta + isin\theta)$$

That is

$$z = \sqrt[n]{\alpha}, r_o^n = r, n\theta_o = \theta + 2k\Pi$$

Thus r_o is the positive nth root of r and $\theta_o = \frac{\theta \pm 2k\Pi}{n}$ has n values for $k = 0, 1, ..., n-1$ all distinct,since increasing k by unity implies increasing the argument by $\frac{2\Pi}{n}$.

The n distinct solutions of (**) are given by

$$(\alpha)^{\frac{1}{n}} = z = r^{\frac{1}{n}}(cos\frac{\theta + 2k\Pi}{n} + isin\frac{\theta + 2k\Pi}{n}) \quad k = 0, 1, ..., n-1 \qquad (***)$$

which are n distinct values of $(\alpha)^{\frac{1}{n}}$

Roots of Unity

A particular example of (**) is when $\alpha = 1$, that is if $z^n = 1, n$ is a positive integer. The roots of the equation are called nth roots of unity.Since

$1 = cos0 + isin0$

then by (***),the nth roots of unity are given by

$$1^{\frac{1}{n}} = (cos\frac{2k\Pi}{n} + isin\frac{2k\Pi}{n}), k = 0, 1, ..., n-1$$

Taking $k = 1$,the root of unity being a complex number and denoted by w is given by

$$w = cos\frac{2\Pi}{n} + isin\frac{2\Pi}{n}$$

Example: Find all the cube roots of -8

Solution:

$$\sqrt[3]{-8} = \sqrt[3]{8(cos\Pi + isin\Pi)}$$

$$= \sqrt[3]{8}(cos\frac{\Pi + 2k\Pi}{3} + isin\frac{\Pi + 2k\Pi}{3}$$

Therefore for

$$k = 0, z_0 = 2(cos\frac{\Pi}{3} + isin\frac{\Pi}{3}) = 1 + i\sqrt{3}$$

$$k = 1, z_1 = 2(cos\Pi + isin\Pi) = -2$$

$$k = 2, z_2 = 2(cos\frac{5\Pi}{3} + isin\frac{5\Pi}{3}) = 1 - i\sqrt{3}$$

Example: Solve $z^4 + 4\sqrt{3} = 4i$

Solution:

$$z^4 + 4\sqrt{3} = 4i \implies \quad z^4 = 4i - 4\sqrt{3}$$

$$\implies \quad z^4 = 8(cos\frac{5\Pi}{6} + isin\frac{5\pi}{6})$$

Hence using De Moivre's theorem

$$z = 8^{\frac{1}{4}}\{cos\frac{\frac{5\Pi}{6} + 2k\Pi}{4} + isin\frac{\frac{5\Pi}{6} + 2k\pi}{4}\} \qquad k = 0, 1, 2, 3$$

The four roots are

$$k = 0 : z_0 = 8^{\frac{1}{4}}(cos\frac{5\Pi}{24} + isin\frac{5\Pi}{24})$$

$$k = 1 : z_1 = 8^{\frac{1}{4}}(cos\frac{17\Pi}{24} + isin\frac{17\Pi}{24}$$

$$k = 2 : z_2 = 8^{\frac{1}{4}}(cos\frac{29\Pi}{24} + isin\frac{29\Pi}{24}) = \overline{z_0}$$

$$k = 3 : z_3 = 8^{\frac{1}{4}}(cos\frac{41\Pi}{24} + isin\frac{41\Pi}{24} = \overline{z_1}$$

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COURSE: MTS 101 ALGEBRA

COURSE UNIT: 3 UNITS
COURSE LECTURER: AGBOOLA A.A.A. [PHD]

COURSE CONTENT: Sets and Binary Operations

1. Elementary Theory of Sets

1.1 Introduction

The concept of set is fundamental in mathematics. A set is a well defined class of objects, such as the set of prime numbers, the set of points on a line, the set of mathematics teachers in a school and so on. The objects making up the set are called the elements, or members of the set. The elements of a set must have some characteristics in common, that is, we must be able to say precisely whether or not an object is a member of a given set. Sets are generally represented by upper case letters A,,B,C,... and their arbitrary members by lower case letters a,b,c,...

If X is a set and x is an element of X, we say that x belongs to X, and we write $x \in X$. If x does not belong to X, we write $x \notin X$. Given a set X and a statement p(x), there is a unique set Y whose elements are precisely those elements $x \in X$ for which p(x) is true. In symbols, we write

$$Y = \{x \in X : \text{p(x) is true}\}.$$

A set X is said to be finite if it has no elements, or if it contains countable number of distinct elements and the process of counting stops at a certain number, say k. The number k is called the cardinality of X, and we write $n(X) = k$. when $k = 0$, set X is said to be empty and X is called a null or a void or an empty set which is denoted by $\emptyset$. A set whose elements are not countable is otherwise called an infinite set.

Definition (1.1.1) Let X and Y be sets. X is said to be a subset of Y if every element of x is an element of Y; that is, if

$$x \in X \Rightarrow x \in Y.$$

If X is a subset of Y, we write $X \subseteq Y$. If Y contains some elements which are not present in X, we say that X is a proper subset of Y, and we write $X \subset Y$. The following statements are evident:

(i) $\emptyset \subset X$.

(ii) $X \subseteq X$.

(iii) If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(iv) $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

The statement given by (iv) is very important: the equality of two sets is usually established by showing that each of the two inclusions is valid.

The term universal or reference set is sometimes used for a set that contains all other sets in a given context and it is represented by $\mathcal{Z}$.

The following sets frequently appear in mathematics:

$$\begin{aligned}
\mathcal{N} &= \{x : \text{x is a natural or counting number}, 1, 2, 3, 4, \ldots\} \\
\mathcal{Z} &= \{x : \text{x is an integer}, \ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathcal{Z}^{+} &= \{x \in \mathcal{Z} : x \geq 0\} \\
\mathcal{Z}^{-} &= \{x \in \mathcal{Z} : x \leq 0\} \\
\mathcal{Q} &= \{x : \text{x is a rational number}, x = \frac{a}{b}, a, b \in \mathcal{Z}, b \neq 0\} \\
\mathcal{Q}^{+} &= \{x \in \mathcal{Q} : x \geq 0\} \\
\mathcal{Q}^{-} &= \{x \in \mathcal{Q} : x \leq 0\} \\
\mathcal{R} &= \{x : \text{x is a real number}\}
\end{aligned}$$

The symbols representing each of the sets should be noted carefully and also it should be observed that

$$\mathcal{N} \subseteq \mathcal{Z} \subseteq \mathcal{Q} \subseteq \mathcal{R}.$$

Definition (1.1.2) Let X and Y be subsets of a reference set $\mathcal{Z}$.

(i) The union of X and Y, written $X \cup Y$, is the set

$$X \cup Y = \{x \in \mathcal{Z} : x \in X \quad \text{or} \quad x \in Y\}.$$

(ii) The intersection of X and Y, written $X \cap Y$, is the set

$$X \cap Y = \{x \in \mathcal{Z} : x \in X \quad \text{and} \quad x \in Y\}.$$

(iii) The difference of X and Y, written $Y - X$, is the set

$$Y - X = \{x \in \mathcal{Z} : x \in Y \quad \text{and} \quad x \notin X\}.$$

If $X \subset Y$, then Y-X is called the complement of X with respect to Y. This is denoted by X' or X^c. X and Y are said to be disjoint if $X \cap Y = \emptyset$. It should be noted carefully that when $x \in X$ or $x \in Y$, it means that x belongs to at least one of X, Y and when $x \in X$ and $y \in Y$, it means that x belongs to both X and Y. The following two statements are immediate:

(i) For any two sets X,Y, $X \cap Y \subset X \subset X \cup Y$.

(ii) If $X \subset W$ and $Y \subset Z$, then $X \cup Y \subset W \cup Z$ and $X \cap Y \subset W \cap Z$.

The formal properties of the operations $\cup$ and $\cap$ are given in the following theorem.

Theorem (1.1.3) Let X, Y, and Z be sets. Then

(i) $X \cup X = X \cap X, \ \forall X$ [idempotent]

(ii) $X \cup Y = Y \cup X$ and $X \cap Y = Y \cap X$ [commutative]

(iii) $X \cup (Y \cup Z) = (X \cup Y) \cup Z$ and $X \cap (Y \cap Z) = (X \cap Y) \cap Z$ [associative]

(iv) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ and $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ [distributive]

Proof: The verifications of (i)-(iii) are easy. To establish (iv), let

$$\begin{aligned}
x \in X \cap (Y \cup Z) &\Leftrightarrow x \in X \quad \text{and} \quad x \in (Y \cup Z) \\
&\Leftrightarrow x \in X \quad \text{and} \quad [x \in Y \quad \text{or} \quad x \in Z] \\
&\Leftrightarrow [x \in X \quad \text{and} \quad x \in Y] \quad \text{or} \quad [x \in X \quad \text{and} \quad x \in Z] \\
&\Leftrightarrow x \in (X \cap Y) \quad \text{or} \quad x \in (X \cap Z) \\
&\Leftrightarrow x \in (X \cap Y) \cup (X \cap Z).
\end{aligned}$$

Also, let

$$\begin{aligned}
x \in X \cup (Y \cap Z) &\Leftrightarrow x \in X \quad \text{or} \quad x \in (Y \cap Z) \\
&\Leftrightarrow x \in X \quad \text{or} \quad [x \in Y \quad \text{and} \quad x \in Z] \\
&\Leftrightarrow [x \in X \quad \text{or} \quad x \in Y] \quad \text{and} \quad [x \in X \quad \text{or} \quad x \in Z] \\
&\Leftrightarrow x \in (X \cup Y) \quad \text{and} \quad x \in (X \cup Z) \\
&\Leftrightarrow x \in (X \cup Y) \cap (X \cup Z).
\end{aligned}$$

Theorem (1.1.4)[De Morgan] Let X and Y be subsets of a universal set $\mathcal{Z}$. Then

(i) $(X \cup Y)^c = X^c \cap Y^c$.

(ii) $(X \cap Y)^c = X^c \cup Y^c$.

Proof: (i) Let

$$\begin{aligned}
x \in (X \cup Y)^c &\Leftrightarrow x \in \mathcal{Z} \quad \text{and} \quad x \notin (X \cup Y) \\
&\Leftrightarrow x \in \mathcal{Z} \quad \text{and} \quad [x \notin X \quad \text{or} \quad x \notin Y] \\
&\Leftrightarrow [x \in \mathcal{Z} \quad \text{and} \quad x \notin X] \quad \text{and} \quad [x \in \mathcal{Z} \quad \text{and} \quad x \notin Y] \\
&\Leftrightarrow x \in X^c \quad \text{and} \quad x \in Y^c \\
&\Leftrightarrow x \in (X^c \cap Y^c).
\end{aligned}$$

(ii) Also let

$$\begin{aligned} x \in (X \cap Y)^c &\Leftrightarrow x \in \mathcal{Z} \quad \text{and} \quad x \notin (X \cap Y) \\ &\Leftrightarrow x \in \mathcal{Z} \quad \text{and} \quad [x \notin X \quad \text{and} \quad x \notin Y] \\ &\Leftrightarrow [x \in \mathcal{Z} \quad \text{and} \quad x \notin X] \quad \text{or} \quad [x \in \mathcal{Z} \quad \text{and} \quad x \notin Y] \\ &\Leftrightarrow x \in X^c \quad \text{or} \quad x \in Y^c \\ &\Leftrightarrow x \in (X^c \cup Y^c). \end{aligned}$$

It is sometimes helpful to illustrate union, intersection, difference and complement of sets by means of Venn diagrams. Circles or ovals are drawn to represent the sets which are enclosed within a rectangle representing the universal set. Venn diagram is a useful tool in establishing basic and simple idenntities involving sets and also in solving two-set and three-set problems.

Theorem (1.1.5) Let X, Y and Z be finite sets contained in the universal set $\mathcal{Z}$. Then $X \cup Y \cup Z$ is finite and

$$n(X \cup Y \cup Z) = n(X) + n(Y) + n(Z) - n(X \cap Y) - n(X \cap Z) - n(Y \cap Z) + n(X \cap Y \cap Z).$$

Proof: Suppose that X, Y and Z are finite sets contained in the universal set $\mathcal{Z}$. Obviously, $X \cup Y \cup Z$ is finite and $\mathcal{Z} = X \cup Y \cup Z$. From the venn diagram, we have

$$\begin{aligned} n(\mathcal{Z}) &= n(X \cup Y \cup Z) \\ &= a + b + c + d + e + f + g \\ &= (a + b + c + d) + (c + d + e + f) + (b + d + f + g) \\ &\quad -(c + d + b + d + f + d) + d \\ &= (a + b + c + d) + (c + d + e + f) + (b + d + f + g) \\ &\quad -[(c + d) + (b + d) + (f + d)] + d \\ &= (a + b + c + d) + (c + d + e + f) + (b + d + f + g) \\ &\quad -(c + d) - (b + d) - (f + d) + d \end{aligned}$$

$$
\begin{aligned}
&= n(X)+n(Y)+n(Z)-n(X\cap Y)-n(X\cap Z) \\
&\quad -n(Y\cap Z)+n(X\cap Y\cap Z) \\
\text{therefore}\quad n(X\cup Y\cup Z) &= n(X)+n(Y)+n(Z)-n(X\cap Y) \\
&\quad -n(X\cap Z)-n(Y\cap Z)+n(X\cap Y\cap Z) \qquad (1)
\end{aligned}
$$

If $Z=\emptyset$, (1) reduces to

$$n(X\cup Y)=n(X)+n(Y)-n(X\cap Y) \qquad (2)$$

Also if X and Y are disjoint, that is, $X\cap Y=\emptyset$, then (2) reduces to

$$n(X\cup Y)=n(X)+n(Y) \qquad (3)$$

Equations (1) and (2) are very useful in dealing with three-set and two-set problems respectively.

1.2 Worked Examples

Example (1.2.1) If A, B and C are subsets of the universal set $\mathcal{Z}$, represent the following sets on venn diagrams:

(a) $(A\cup B)\cap(A\cup C)$;

(b) $(A\cap B)\cup(A\cap C)$;

(c) $A\cup(B\cap C)$;

(d) $A\cap(B\cup C)$.

What do you notice about:

(i) (a) and (c) ? (ii) (b) and (d) ?

Solution:

Example (1.2.2) The universal set $\mathcal{Z}$ is the set of all integers. A,B and C are subsets of $\mathcal{Z}$

defined as follows:

$$\begin{aligned} A &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \\ B &= \{x : 0 \leq x \leq 9\} \\ C &= \{x : -4 < x \leq 0\} \end{aligned}$$

(a) Write down the sets A', where A' is the complement of A with respect to $\mathcal{Z}$.

(b) Find $B \cap C$.

(c) Find the members of the sets $B \cup C, A \cap B$ and $A \cap C$ and hence show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Solution: (a)

$$\begin{aligned} A' &= \{x : x \in \mathcal{Z} \quad \text{and} \quad x \notin A\} \\ &= \{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\} \end{aligned}$$

(b)

$$\begin{aligned} B &= \{x : 0 \leq x \leq 9\} \\ &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ C &= \{x : -4 < x \leq 0\} \\ &= \{-3, -2, -1, 0\} \\ B \cap C &= \{0\}. \end{aligned}$$

(c)

$$\begin{aligned} B \cup C &= \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ A \cap B &= \{0, 2, 4, 6, 8\} \\ A \cap C &= \{-2, 0\} \\ (A \cap B) \cup (A \cap C) &= \{-2, 0, 2, 4, 6, 8\} \\ A \cap (B \cup C) &= \{-2, 0, 2, 4, 6, 8\} = (A \cap B) \cup (A \cap C). \end{aligned}$$

Example (1.2.3) In a certain class, 22 pupils take one or more of Chemistry, Economics and Government. 12 take Economics (E), 8 take Government (G) and 7 take Chemistry (C). Nobody takes Economics and Chemistry and 4 pupils take Economics and Government.

(a) (i) Using set notation and the letters indicated above, write down the two statements in the last sentence

(ii) Draw a venn diagram to illustrate the information.

(b) How many pupils take

(i) both Chemistry and Government ?

(ii) Government only ?

Solution: (a) (i) Nobody takes Economics and Chemistry :

$$E \cap C = \emptyset \Rightarrow \quad n(E \cap C) = 0.$$

4 pupils take Economics and Government:

$$n(E \cap G) = 4.$$

(ii) Since $E \cap C = \emptyset$, it follows that $E \cap C \cap G = \emptyset$.

Given that $n(E \cup G \cup C) = 22$, we have

$$\begin{aligned} 8 + 4 + 4 - x + x + 7 - x &= 22 \\ \therefore \quad x &= 1. \end{aligned}$$

(b) From the venn diagram, we have that

(i) One pupil takes both Chemistry and Government.

(ii) Three pupils take Government only.

Example (1.2.4) A school has 37 vacancies for teachers, out of which 22 are for English Language, 20 for History and 17 for Fine Art. Of these vacancies 11 are for both English Language and History, 8 for both History and Fine Art and 7 for English Language and Fine Art.

Using a venn diagram, find the number of teachers who must be able to teach:

(a) all the three subjects;

(b) Fine Art only;

(c) English Language and History, but not Fine Art.

Solution: Let English Language, History and Fine Art be represented by E, H and F respectively. From the given data we have

$$\begin{aligned} n(E \cup H \cup F) &= 37 \\ n(E \cap H) &= 11 \\ n(H \cap F) &= 8 \\ n(E \cap F) &= 7. \end{aligned}$$

Let $n(E \cap H \cap F) = x$, then

$$\begin{aligned} n(E \cup H \cup F) &= 4 + x + 7 - x + x + 11 - x + 1 + x + 8 - x + 2 + x = 37 \\ \Rightarrow \quad 33 + x &= 37 \\ \therefore \quad x &= 4. \end{aligned}$$

It is clear from the venn diagram that

(a) 4 teachers must be able to teach all the three subjects.

(b) 6 teachers must be able to teach Fine Art only.

(c) 20 teachers must be able to teach English Language and History, but not Fine Art.

Example (1.2.5) (a) In a certain school, 3 teachers teach Further Mathematics and 6 teach General Mathematics. If there are 7 Mathematics teachers in the school, how many of them teach both Futher Mathematics and General Mathematics ?

(b) A newsagent sell three papers, the Guardian, the Punch and the Tribune. 70 customers buy the Guardian, 80 the Punch, and 90 the Tribune. 20 buy both the Guardian and the Punch, 25 the Punch and the Tribune, and 30 the Guardian and the Tribune. If 15 customers buy all the three papers, how many customers has the newsagent?

Solution: (a) Let F and G represent the sets of Further Mathematics and General Mathematics teachers respectively. From the given data we have $n(F) = 3, n(G) = 6, n(F \cup G) = 7$. We are to find $n(F \cap G)$. Using the formula for a two-set problem, we have

$$\begin{aligned} n(F \cup G) &= n(F) + n(G) - n(F \cap G) \\ \Rightarrow \quad 7 &= 3 + 6 - n(F \cap G) \\ \therefore \quad n(F \cap G) &= 9 - 7 \\ &= 2. \end{aligned}$$

Hence, 2 teachers teach both Further Mathematics and General Mathematics.

(b) Let G, P and T represent the set of customers who buy the Guardian, the Punch and the Tribune newspapers respectively. From the given data we have $n(G) = 70, n(P) = 80, n(T) = 90, n(G \cap P) = 20, n(P \cap T) = 25, n(G \cap T) = 30$ and $n(G \cap P \cap T) = 15$ It is required to find $n(G \cup P \cup T)$.

Using the formula for the three-set problem, we have

$$
\begin{aligned}
n(G \cup P \cup T) &= n(G) + n(P) + n(T) - n(G \cap P) - n(P \cap T) - n(G \cap T) + n(G \cap P \cap T) \\
&= 70 + 80 + 90 - 20 - 25 - 30 + 15 \\
7 = {}& 255 - 75 \\
&= 180.
\end{aligned}
$$

Thus the newsagent has 180 customers.

The information is represented in the venn diagram below.

Example (1.2.6) If A and B are subsets of the reference set X, use set theoretic argument to show that:

(a) $(A^c)^c = A$,

(b) $X - A^c = X \cap A$,

(c) $A \cup (A^c \cap B) = A \cup B$,

(d) $A \cap (A^c \cup B) = A \cap B$.

Solution: (a) Let

$$
\begin{aligned}
x \in (A^c)^c &\Leftrightarrow x \notin A^c \\
&\Leftrightarrow x \in A \\
\therefore \quad (A^c)^c &= A.
\end{aligned}
$$

(b) Let

$$x \in (X - A^c) \Leftrightarrow x \in X \quad \text{and} \quad x \notin A^c$$

$$
\begin{aligned}
& \Leftrightarrow \quad x \in X \quad \text{and} \quad x \in A \\
& \Leftrightarrow \quad x \in X \cap A \\
\therefore \quad X - A^c \; & = \quad X \cap A.
\end{aligned}
$$

(c) Let

$$
\begin{aligned}
x \in A \cup (A^c \cap B) & \Leftrightarrow \quad x \in A \quad \text{or} \quad x \in (A^c \cap B) \\
& \Leftrightarrow \quad x \in A \quad \text{or} \quad [x \in A^c \quad \text{and} \quad x \in B] \\
& \Leftrightarrow \quad x \in A \quad \text{or} \quad x \in B \\
& \Leftrightarrow \quad x \in (A \cup B) \\
\therefore \quad A \cup (A^c \cap B) & = \quad A \cup B.
\end{aligned}
$$

(c) Let

$$
\begin{aligned}
x \in A \cap (A^c \cup B) & \Leftrightarrow \quad x \in A \quad \text{and} \quad x \in (A^c \cup B) \\
& \Leftrightarrow \quad x \in A \quad \text{and} \quad [x \in A^c \quad \text{or} \quad x \in B] \\
& \Leftrightarrow \quad x \in A \quad \text{and} \quad x \in B \\
& \Leftrightarrow \quad x \in (A \cap B) \\
\therefore \quad A \cap (A^c \cup B) & = \quad A \cap B.
\end{aligned}
$$

1.3 Self Assessment Problems

(1.3.1) If A, B, C are subsets of X such that

$$
\begin{aligned}
X &= \{\alpha, \beta, \gamma, \lambda, \psi, \theta, \omega, \tau, \mu, \nu\} \\
A &= \{\alpha, \beta, \lambda, \theta, \omega, \tau, \mu\} \\
B &= \{\alpha, \beta, \gamma, \lambda, \omega, \mu\} \\
C &= \{\beta, \gamma, \lambda, \omega, \mu\}
\end{aligned}
$$

Find:

(a) A'

(b) B'

(c) C'

(d) $A' \cap B' \cap C$

(e) $A \cap B' \cap C'$

(f) $A' \cap B \cap C'$

(g) $A' \cap B' \cap C'$

and show that

(i) $(C \cup B) \cap A = (C \cap A) \cup (B \cap A)$

(ii) $(C \cap B) \cup A = (C \cup A) \cap (B \cup A)$.

Answer:

$$\begin{aligned}
A' &= \{\gamma, \psi, \nu\} \\
B' &= \{\psi, \theta, \tau, \nu\} \\
C' &= \{\alpha, \psi, \theta, \tau, \nu\} \\
A' \cap B' \cap C &= \{\} = \emptyset \\
A \cap B' \cap C' &= \{\theta, \tau\} \\
A' \cap B \cap C' &= \{\} = \emptyset \\
A' \cap B' \cap C' &= \{\psi, \nu\}
\end{aligned}$$

(1.3.2) The universal set $\mathcal{Z}$ is the set of all integers and the subsets P, Q, R of $\mathcal{Z}$ are given by

$$\begin{aligned}
P &= \{x : x \leq 0\} \\
Q &= \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} \\
R &= \{x : -2 \leq x < 7\}.
\end{aligned}$$

(a) Find $Q \cap R$

(b) Find R' where R' is the complement of R with respect to $\mathcal{Z}$.

(c) Find $P' \cap R'$.

(d) List the members of $(P \cap Q)'$.

Answer:

$$\begin{aligned}
Q \cap R &= \{-1, 1, 3, 5\} \\
R' &= \{\ldots, -7, -6, -5, -4, -3, 7, 8, \ldots\} \\
P' \cap R' &= \{7, 8, 9, 10, \ldots\} \\
(P \cap Q)' &= \{\ldots, -6, -4, -2, 0, 1, 2, 3, 4, 5, 6, \ldots\}
\end{aligned}$$

(1.3.3) In a survey of 290 newspaper readers, 181 of them read the Daily Times, 142 read Guardian, 117 read the Punch and each reads at least one of the three papers. If 75 read the Daily Times and the Guardian, 60 read the Daily Times and the Punch and 54 read the Guardian and the Punch.

(a) Draw a venn diagram to illustrate this information.

(b) How many readers read

(i) all three papers,

(ii) exactly two of the papers,

(iii) exactly one of the papers,

(iv) the Guardian alone ?

Answer: (a) (b) (i) 39 (ii) 72 (iii) 179 (iv) 52

(1.3.4) After the registration of 100 freshmen, the following statistics were revealed: 60 were taking Mathematics, 44 were taking Physics, 30 were taking Chemistry, 15 were taking Physics and Chemistry, 6 were taking both Mathematics and Physics but not Chemistry, 24 were taking Mathematics and Chemistry, and 10 were taking all three subjects.

(a) Show that 54 were enrolled in only one of the three subjects.

(b) Show that 35 were enrolled in at least two of them.

(1.3.5) Of a sample of 1000 students surveyed at the end of a term, 100 had applied for Lagos University, 80 for Ife, 75 for Benin, 30 had applied for Lagos and Benin, 20 for Lagos and Ife, 15 for Benin and Ife, and 5 had applied for all the three universities. Show that

(a) 195 had applied for at least one of the universities.

(b) 805 had not applied.

(c) 55 had applied for Lagos only.

(d) 35 had applied for Benin only.

(e) 50 had applied for Ife only.

(1.3.6) A, B, C are subsets of a universal set X. Show that:

(a) $A \cup (A \cap B) = A = A \cap (A \cup B)$

(b) $(A \cup B') \cap (A \cup B) = A$

(c) $A \cup (B \cap C) \neq (A \cap B) \cup C$

(1.3.7) A, B, C are subsets of a universal set X. Show that:

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

(c) $(A')' = A$.

Hence or otherwise, show that

(d) $(A \cup B')' = A' \cap B$

(e) $(A' \cap B)' = A \cup B'$

(f) $(A' \cup B')' = A \cap B$

(g) $(A' \cap B')' = A \cup B$.

(1.3.8) A, B, C are subsets of a universal set X. Show that:

(a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(b) $(A - B) - (A - C) = A \cap (C - B)$

(c) $(A - B) \cup (A - C) = A - (B \cap C)$

(d) $(A - B) \cap (A - C) = A - (B \cup C)$.

2. Binary Operations

2.1 Introduction

Due to our familiarity with the four basic arithmetic operations of addition, subtraction, multiplication and division, if we are asked to add 2 and 3, our answer will be 5 and not 1 which could have been if we are to add in modulo 4. This shows that our answer depends on the rule of combination of the numbers involved.

A binary operation is said to have been performed if two elements of a set are combined according to a well defined rule to produce another element of the set. For example, $2, 3 \in \mathcal{N}$ and $2 + 3 = 5 \in \mathcal{N}$. Binary operations are often denoted by the symbols $\oplus, \otimes, \circ, \nabla, \diamond, *$ and so on.

Example (2.1.1) A binary operation $*$ is defined over $\mathcal{R}$ the set of real numbers by

$$x * y = x + y + xy.$$

(a) Evaluate:

$$(i) \quad 2 * 3 \qquad (ii) \quad 3 * 2 \qquad (iii) \quad 2 * (3 * 4) \qquad (iv) \quad (2 * 3) * 4.$$

(b) Solve the equations:

$$(i) \quad x * 3 = 19 \qquad (ii) \quad (x * 3) + (2 * x) = 40 \qquad (iii) \quad x * x = 48.$$

Solution: (a) (i) Given that $x * y = x + y + xy$, then

$$\begin{aligned} 2 * 3 &= 2 + 3 + 2 \times 3 \\ &= 5 + 6 \\ &= 11. \end{aligned}$$

(ii)

$$\begin{aligned} 3 * 2 &= 3 + 2 + 3 \times 2 \\ &= 5 + 6 \\ &= 11. \end{aligned}$$

(iii)

$$\begin{aligned} 2 * (3 * 4) &= 2 * (3 + 4 + 3 \times 4) \\ &= 2 * 19 \\ &= 2 + 19 + 2 \times 19 \\ &= 59. \end{aligned}$$

(iv)

$$\begin{aligned} (2 * 3) * 4 &= 11 * 4 \\ &= 11 + 4 + 11 \times 4 \\ &= 59. \end{aligned}$$

(b) (i)

$$\begin{aligned} x * 3 &= 19 \\ \Rightarrow \quad x + 3 + 3x &= 19 \\ \Rightarrow \quad 4x &= 16 \\ \therefore \quad x &= 4. \end{aligned}$$

(ii)

$$\begin{aligned} (x * 3) + (2 * 3) &= 40 \\ \Rightarrow \quad x + 3 + 3x + 2 + x + 2x &= 40 \\ \Rightarrow \quad 7x &= 35 \\ \therefore \quad x &= 5. \end{aligned}$$

(iii)

$$\begin{aligned} x * x &= 48 \\ \Rightarrow \quad x + x + x^2 &= 48 \\ \Rightarrow \quad x^2 + 2x - 48 &= 0 \\ \Rightarrow \quad (x-6)(x+8) &= 0 \\ \Leftrightarrow \quad x &= 6 \quad \text{or} \quad x = -8. \end{aligned}$$

2.2 Properties of Binary Operations

Closure Property: A set A is said to be closed under a binary operation $*$ if for all $a, b \in A$, $a * b \in A$. For example, $\mathcal{N}$ is closed under the usual addition and multiplication.

Example (2.2.1) Let $X = \{x : 1 \leq x \leq 4\}$ and let a binary operation $*$ be defined on X such that for every $x, y \in X$,

$$x * y = x \times_2 y$$

where $\times_2$ is the multiplication in modulo 2. Examine whether or not X is closed under $*$.

Solution: Given that

$$\begin{aligned} X &= \{x : 1 \leq x \leq 4\} \\ &= \{1, 2, 3, 4\}, \end{aligned}$$

Consider the table below.

$\times_2$	1	2	3	4
1	1	0	1	0
2	0	0	0	0
3	1	0	1	0
4	0	0	0	0

It is clear from the table that X is not closed under $*$ since $0 \notin X$. **Commutative Property:** A binary operation $*$ over a set A is said to be commutative if for all $a, b \in A$, $a * b = b * a$. For example, the ordinary addition and multiplication are commutative over $\mathcal{Q}$ the set of rationals but the operations of ordinary subtraction and division are anti-commutative over $\mathcal{Q}$.

Associative Property: A binary operation $*$ over a set A is said to be associative if for all $a, b, c \in A$,

$$a * (b * c) = (a * b) * c.$$

For example, the operations of ordinary addition and multiplication are associative over $\mathcal{R}$ the set of real numbers but the operations of ordinary subtraction and division are not.

Distributive Property: A binary operation $*$ over a set A is said to be distributive over another binary operation $*'$ also defined over A if for all $a, b, c \in A$,

$$a * (b *' c) = (a * b) *' (a * c).$$

For example, over $\mathcal{Z}$ the set of integers, the operation of usual multiplication is distributive over the operation of ordinary addition.

Also, the two operations $\cup$ and $\cap$ are distributive over each other since for every set A, B, C,

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C). \end{aligned}$$

Example (2.2.2) Let $\times_6$ denotes multiplication modulo 6 and let $+_6$ denotes addition modulo 6. Show that $\times_6$ is distributive over $+_6$.

Solution: Let us pick 8, 11, and 15 from the set of natural numbers. By definition, $\times_6$ will be distributive over $+_6$ if and only if

$$a \times_6 (b +_6 c) = (a \times_6 b) +_6 (a \times_6 c) \qquad \forall a, b, c \in \mathcal{N}.$$

Now setting $a = 8, b = 11, c = 15$, we have

$$\begin{aligned} LHS &= a \times_6 (b +_6 c) \\ &= 8 \times_6 (11 +_6 15) \\ &= 8 \times_6 2 \\ &= 4\text{mod}6. \end{aligned}$$

Also put $a = 8, b = 11, c = 15$, we have

$$\begin{aligned} RHS &= (a \times_6 b) +_6 (a \times_6 c) \\ &= (8 \times_6 11) +_6 (8 \times_6 15) \\ &= 4 +_6 0 \\ &= 4\text{mod}6 \\ &= LHS. \end{aligned}$$

Thus, $\times_6$ is distributive over $+_6$. The reader should also try the verification using other natural numbers.

Identity Element: Let $*$ be a binary operation over a set A. An element $e \in A$ is said to be an identity or a neutral element if for all $a \in A$,

$$a * e = e * a = a.$$

For example in $\mathcal{R}$, the set of real numbers, 0 and 1 are the additive and multiplicative identities respectively since

$$\begin{aligned} 0 + a &= a + 0 = a \\ 1 \times b &= b \times 1 = b \quad \forall\, a, b \in \mathcal{R}. \end{aligned}$$

Inverse Element: Let $*$ be a binary operation over a set A. An element $b \in A$ is said to be an inverse of an element $a \in A$ if for all $a \in A$,

$$a * b = b * a = e$$

where e is the identity element of the set A. The inverse of a if it exists is generally denoted by a^{-1}. For example in $\mathcal{R}$, the set of real numbers,

$$a + (-a) = (-a) + a = 0 \quad \forall\, a \in \mathcal{R}$$

also for all $(a \neq 0) \in \mathcal{R}$,

$$a \times a^{-1} = a^{-1} \times a = 1.$$

Thus, (-a) is the additive inverse of an element $a \in \mathcal{R}$ and a^{-1} is the multiplicative of a nonzero element $a \in \mathcal{R}$.

It should be noted that the identity element and the inverse element in a set under a given binary operation are unique. Also, the existence of an inverse element depends on the existence of an identity element in a given set under a given binary operation.

2.3 Worked Examples

Example (2.3.1) A binary operation $*$ is defined on $\mathcal{R}$ the set of real numbers by

$$x * y = \frac{xy}{x+y}, \quad x + y \neq 0.$$

(a) Show that $*$ is commutative and associative.

(b) Obtain the values of:

$$(i) \quad 3 * -5 \qquad (ii) \quad 2 * 7 \qquad (iii) \quad -3 * (5 * -7) \qquad (iv) \quad \frac{2}{3} * \frac{-5}{7}.$$

(c) Obtain the values of x for which

$$(i) \quad 3 * x = 5/7 \qquad\qquad (ii) \quad (x * 3) + (4 * x) = 7.$$

Solution: (a)

$$\begin{aligned} x * y &= \frac{xy}{x+y} \\ &= \frac{yx}{y+x} \\ &= y * x. \end{aligned}$$

The commutativity of $*$ follows.

For associativity, we must show that for all $x, y, z \in \mathcal{R}$, $x * (y * z) = (x * y) * z$. Now put

$$\begin{aligned} LHS &= x * (y * z) \\ &= x * \frac{yz}{y+z} \\ &= \frac{xyz}{y+z} \div \left(x + \frac{yz}{y+z} \right) \\ &= \frac{xyz}{y+z} \times \frac{y+z}{xy+xz+yz} \\ &= \frac{xyz}{xy+xz+yz}. \\ RHS &= (x * y) * z \\ &= \frac{xy}{x+y} * z \\ &= \frac{xyz}{x+y} \div \left(\frac{xy}{x+y} + z \right) \\ &= \frac{xyz}{x+y} \times \frac{x+y}{xy+xz+yz} \\ &= \frac{xyz}{xy+xz+yz} \\ &= LHS. \end{aligned}$$

The associativity of $*$ then follows.

(b) (i)

$$\begin{aligned} 3 * -5 &= \frac{3 \times -5}{3-5} \\ &= \frac{15}{2}. \end{aligned}$$

(ii)

$$\begin{aligned} 2 * 7 &= \frac{2 \times 7}{2+7} \\ &= \frac{14}{9}. \end{aligned}$$

(iii)

$$\begin{aligned}
-3 * (5 * -7) &= -3 * \left(\frac{5 \times -7}{5 - 7}\right) \\
&= -3 * \frac{35}{2} \\
&= \frac{-3 \times \frac{35}{2}}{-3 + \frac{35}{2}} \\
&= -\frac{105}{29}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\frac{2}{3} * \frac{-5}{7} &= \frac{\frac{2}{3} \times \frac{-5}{7}}{\frac{2}{3} - \frac{5}{7}} \\
&= \frac{-10}{21} \div \frac{-1}{21} \\
&= \frac{-10}{21} \div \frac{-21}{1} \\
&= 10.
\end{aligned}$$

(c) (i)

$$\begin{aligned}
x * y &= \frac{5}{7} \\
\Rightarrow \quad \frac{3x}{3 + x} &= \frac{5}{7} \\
\Rightarrow \quad 21x - 5x &= 15 \\
\therefore \quad x &= \frac{15}{16}.
\end{aligned}$$

(ii)

$$\begin{aligned}
(x * 3) + (4 * x) &= 7 \\
\Rightarrow \quad \frac{3x}{3 + x} + \frac{4x}{4 + x} &= 7 \\
\Rightarrow \quad 12x + 3x^2 + 4x^2 + 12x &= 7(x^2 + 7x + 12) \\
\Rightarrow \quad 49x - 24x &= -84 \\
\therefore \quad x &= -\frac{84}{25}.
\end{aligned}$$

Example (2.3.2) The operation $*$ is defined over $\mathcal{R}$ the set of real numbers by

$$p * q = p + q - \frac{1}{2}pq.$$

(a) Show that $*$ is commutative and associative.

(b) Find the identity element for the operation $*$.

(c) Find the inverse (under $*$) of the real number p, stating any value of p for which no inverse exists.

(d) Determine whether or not

$$p * (q + r) = (p * q) + (p * r), \qquad \forall\ p, q, r \in \mathcal{R}.$$

Solution: (a)

$$\begin{aligned} p * q &= p + q - \frac{1}{2}pq \\ &= q + p - \frac{1}{2}qp \\ &= q * p. \end{aligned}$$

The commutativity of $*$ is immediate.

For associativity, let $p, q, r \in \mathcal{R}$. Then

$$\begin{aligned} p * (q * r) &= p * (q + r - \frac{1}{2}qr) \\ &= p + (q + r - \frac{1}{2}qr) - \frac{1}{2}p(q + r - \frac{1}{2}qr) \\ &= p + q + r - \frac{1}{2}qr - \frac{1}{2}pq - \frac{1}{2}pr + \frac{1}{4}pqr. \\ (p * q) * r &= (p + q - \frac{1}{2}pq) * r \\ &= (p + q - \frac{1}{2}pq) + r - \frac{1}{2}(p + q - \frac{1}{2}pq)r \\ &= p + q + r - \frac{1}{2}qr - \frac{1}{2}pq - \frac{1}{2}pr + \frac{1}{4}pqr. \\ &= p * (q * r). \end{aligned}$$

The associativity of $*$ is immediate.

(b) Suppose that $e \in \mathcal{R}$ is the identity element for $*$, then

$$\begin{aligned} p * e &= e * p = p \\ \Rightarrow \quad p + e - \frac{1}{2}pe &= p \\ \Rightarrow \quad e(2 - p) &= 0 \\ \Leftrightarrow \quad e &= 0 \qquad \text{[provided that } p \neq 2]. \end{aligned}$$

(c) Suppose that k is the inverse of p under $*$, then

$$\begin{aligned} p * k &= k * p = e = 0 \\ \Rightarrow \quad p + k - \frac{1}{2}pk &= 0 \\ \Rightarrow \quad k(p - 2) &= 2p \\ \therefore \quad k &= \frac{2p}{p - 2}. \end{aligned}$$

No inverse will exist if $p-2=0$ that is when $p=2$.

(d)

$$\begin{aligned}
p*(q+r) &= p+(q+r)-\frac{1}{2}p(q+r) \\
&= p+q+r-\frac{1}{2}pq-\frac{1}{2}pr. \\
(p*q)+(p*r) &= p+q-\frac{1}{2}pq+p+r-\frac{1}{2}pr \\
&= 2p+qr-\frac{1}{2}pq-\frac{1}{2}pr \\
&\neq p*(q+r).
\end{aligned}$$

Example (2.3.3) Let X be a nonempty set with associative binary operation $\triangle$. Let $x,y,z\in X$. Suppose x commutes with y and z, show that x commutes also with $y\triangle z$.

Solution: Given that x commutes with y and z, then $x\triangle y=y\triangle x$ and $x\triangle z=z\triangle x$. Now,

$$\begin{aligned}
x\triangle(y\triangle z) &= (x\triangle y)\triangle z \qquad \text{[since } \triangle \text{ is associative]} \\
&= (y\triangle x)\triangle z \\
&= y\triangle(x\triangle z) \\
&= y\triangle(z\triangle x) \\
&= (y\triangle z)\triangle x.
\end{aligned}$$

Evidently, x commutes with $y\triangle z$.

Example (2.3.4) Let J be a nonempty set with associative binary operation $\triangle$. Show that the binary operation ∇ given by

$$x\nabla y=x\triangle j\triangle y$$

is also associative. If $\triangle$ is commutative, is ∇ commutative ?

Solution: Given that $\triangle$ is associative binary operation, let $x,y,z\in J$. Then

$$\begin{aligned}
x\nabla(y\nabla z) &= x\nabla(y\triangle j\triangle z) \\
&= x\triangle j\triangle(y\triangle j\triangle z) \\
&= x\triangle j\triangle y\triangle j\triangle z. \\
(x\nabla y)\nabla z &= (x\triangle j\triangle y)\nabla z \\
&= (x\triangle j\triangle y)\triangle j\triangle z \\
&= x\triangle j\triangle y\triangle j\triangle z \\
&= x\nabla(y\nabla z).
\end{aligned}$$

Accordingly, ∇ is associative over J.

Lastly, suppose that $\triangle$ is commutative, then ∇ will be commutative if and only if for all $x, y \in J$, we can show that $x\nabla y = y\nabla x$. To this end,

$$\begin{aligned} x\nabla y &= x\triangle j\triangle y \\ &= x\triangle(j\triangle y) \\ &= x\triangle(y\triangle j) \\ &= (x\triangle y)\triangle j \\ &= (y\triangle x)\triangle j \\ &= y\triangle(x\triangle j) \\ &= y\triangle(j\triangle x) \\ &= y\triangle j\triangle x \\ &= y\nabla x. \end{aligned}$$

The required result then follows.

Example (2.3.5) Consider the set I of ordered pairs

$$I = \{(m, n) : \text{m, n are natural numbers}\}.$$

An operation $\oplus$ is defined on I by

$$(a, b) \oplus (c, d) = (a + c, b + d).$$

Show that this operation is commutative and associative.

Any two elements (a,b), (c,d) in I are to be considered equal if and only if $a+d = b+c$. Show that any element of the form (n,n) may be regarded as a neutral element with respect to $\oplus$.

Given that (r,s) is an inverse of (p,q), find the relationship between p, q, r, s. Hence find an inverse for the element (7,5) and an inverse for the element (m,n).

Solution: Let (a,b), (c,d) and (e,f) be any elements of I. Then

$$\begin{aligned} (a, b) \oplus (c, d) &= (a + c, b + d) \\ &= (c + a, d + b) \\ &= (c, d) \oplus (a, b). \end{aligned}$$

This shows that $\oplus$ is commutative.

Also,

$$(a, b) \oplus ((c, d) \oplus (e, f)) = (a, b) \oplus (c + e, d + f)$$

$$\begin{aligned}
&= (a+(c+e), b+(d+f)) \\
&= ((a+c)+e, (b+d)+f) \\
&= (a+c, b+d) \oplus (e,f) \\
&= ((a,b) \oplus (c,d)) \oplus (e,f).
\end{aligned}$$

This establishes the associativity of $\oplus$.

Next, given that $(a,b) = (c,d)$ if and only if $a+d = b+c$, let $(e,f) \in I$ be an identity element. Then for any $(a,b) \in I$,

$$\begin{aligned}
(a,b) \oplus (e,f) &= (e,f) \oplus (a,b) = (a,b) \\
\Rightarrow \quad (a+e, b+f) &= (a,b) \\
\Leftrightarrow \quad a+e+b &= b+f+a \\
\Leftrightarrow \quad e &= f = n.
\end{aligned}$$

Hence, any element of the form (n,n) where n is a natural number is a neutral element with respect to $\oplus$.

Lastly, given that (r,s) is an inverse of (p,q), then

$$\begin{aligned}
(r,s) \oplus (p,q) &= (p,q) \oplus (r,s) = (n,n) \\
\Rightarrow \quad (p+r, q+s) &= (n,n) \\
\Leftrightarrow \quad p+r+n &= q+s+n \\
\Leftrightarrow \quad p+r &= q+s \qquad (1)
\end{aligned}$$

which is the required relationship between p, q, r and s.
To obtain the inverse of (7,5), put $(p,q) = (7,5)$ in equation (1) to obtain

$$7 + r = 5 + s$$

This equation is obviously satisfied by $r = 5$ and $s = 7$. Hence, (5,7) is the inverse of (7,5). Using the same procedure, we obtain (n,m) as the inverse of (m,n).

Example (2.3.6) Two binary operations $\oplus$ and $\otimes$ over the universal set $\mathcal{Z}$ are defined by

$$\begin{aligned}
A \oplus B &= A \cup B, \\
A \otimes B &= A \cap B \quad \forall\, A, B \subset \mathcal{Z}.
\end{aligned}$$

Show that:

(a) $\oplus$ is both commutative and associative,

(b) $\otimes$ is both commutative and associative,

(c) $\oplus$ is distributive over $\otimes$,

(d) $\otimes$ is distributive over $\oplus$.

Solution: For every $A, B, C \subset \mathcal{Z}$, we have

(a)

$$\begin{aligned} A \oplus B &= A \cup B \\ &= B \cup A \\ &= B \oplus A. \\ A \oplus (B \oplus C) &= A \cup (B \cup C) \\ &= (A \cup B) \cup C \\ &= (A \oplus B) \oplus C. \end{aligned}$$

(b)

$$\begin{aligned} A \otimes B &= A \cap B \\ &= B \cap A \\ &= B \otimes A. \\ A \otimes (B \otimes C) &= A \cap (B \cap C) \\ &= (A \cap B) \cap C \\ &= (A \otimes B) \otimes C. \end{aligned}$$

(c)

$$\begin{aligned} A \oplus (B \otimes C) &= A \cup (B \cap C) \\ &= (A \cup B) \cap (A \cup C) \\ &= (A \oplus B) \otimes (A \oplus C). \end{aligned}$$

(d)

$$\begin{aligned} A \otimes (B \oplus C) &= A \cap (B \cup C) \\ &= (A \cap B) \cup (A \cap C) \\ &= (A \otimes B) \oplus (A \otimes C). \end{aligned}$$

2.4 Self Assessment Problems

(2.4.1) A binary operation $*$ is defined over the set $\mathcal{R}$ of real numbers by

$$x * y = x + y - x^2 y.$$

(a) Determine whether or not $*$ is commutative and associative.

(b) Evaluate:

$$(i) \quad 2 * 3 \qquad (ii) \quad -5 * 4 \qquad (iii) \quad 3 * (4 * 5).$$

(c) Find the value(s) of x for which:

$$(i) \quad 4 * x = 34 \qquad (ii) \quad (3 * x) + (x * 3) = 8.$$

Answer:

(b) -7, -101, 571

(c) -2, -1/3 or -2.

(2.4.2) The function f is defined by

$$f : x \to 3x - 2, \qquad x \in \mathcal{R}.$$

(a) The binary operation $\circ$ on the set $\mathcal{R}$ is such that

$$f(p \circ q) = f(p) \times f(q) \qquad \forall\, p, q \in \mathcal{R}.$$

(i) Show that $p \circ q = 3pq - 2p - 2q + 2$.

(ii) Show that $\circ$ is commutative and associative.

(iii) Find the identity element for the operation.

(iv) Find the inverse (under $\circ$) of the real number p, stating any value of p for which no inverse exists.

(b) Another binary operation $\bullet$ on the set $\mathcal{R}$ is such that

$$f(p \bullet q) = \frac{f(p)}{f(q)}, \quad f(q) \neq 0 \quad \forall\, p, q \in \mathcal{R}.$$

(i) Show that

$$p \bullet q = \frac{p + 2q - 2}{3q - 2}, \qquad q \neq \frac{2}{3}.$$

(ii) Show that $\bullet$ is neither commutative nor associative.

(iii) Determine whether or not

$$p \bullet (q \circ r) = (p \bullet q) \circ (p \bullet r) \quad \forall\, p, q, r \in \mathcal{R}.$$

Answer:

(a) (iii) 1 (iv) $\frac{2p-1}{3p-2}$, $p = 2/3$.

(2.4.3) Let S be the set of all ordered pairs $x = (x_1, x_2)$ with x_1 and x_2 real numbers. A binary operation $*$ is defined on S by

$$a * b = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1).$$

Show that this operation is commutative and associative.

Determine the identity element for this operation, and also the inverse of any element x. Hence solve:

$$a * x = b \qquad \text{where} a = (3, 4), b = (5, 6).$$

Answer:

identity $= (1, 0)$, $x^{-1} = \left(\frac{x_1}{x_1^2+x_2^2}, \frac{-x_2}{x_1^2+x_2^2}\right)$, $x = \left(\frac{39}{25}, \frac{-2}{25}\right)$.

(2.4.4) Let X be a nonempty set with associative binary operation $\circ$. If e and f are elements of X such that $x \circ e = x$ and $f \circ x = x$ for all x in X, show that $e = f$.

Furthermore, if $x \circ y = e = z \circ x$, show that $y = z$.

(2.4.5) For any two subsets X and Y of a universal set $\mathcal{Z}$, the operation $\bullet$ is defined by

$$X \bullet Y = (X \cap Y') \cup (Y \cap X'),$$

where X', Y' denote the complements of X and Y respectively. Show that:

(a) the operation is commutative;

(b) the empty set $\emptyset$ is the identity element for $\bullet$;

(c) every element is its own inverse.

(2.4.6) Find the identity element, if it exists, and the inverse of 5 when each of the following operations is defined on $\mathcal{R}$ the set or real numbers.

(a) $p * q = p + q$

(b) $p * q = pq$

(c) $p * q = p + q + pq$

(d) $p * q = pq + 2p + 2q$

(e) $p * q = \sqrt{pq}$

(f) $p * q = \frac{p}{q} + \frac{q}{p}$

(g) $p * q = \frac{p}{q} - p$. Answer:

(a) 0,-5 (b) 1,1/5 (c) 0,-5/6 (d) no identity (e) no identity (f) no identity (g) 1/2,5/3.