MTS 102 LECTURE NOTE
FUNCTIONS, LIMITS AND CONTINUOUS FUNCTION
BY
A. D AKWU

Functions of a real variable

(1) Function: Let x and y be real number, if there exist a relation between x and y such that x is given, then y is determined, we say that y is a function of x and x is called independent variable and y is the dependent variable, that is $y = f(x)$.
For example: I f $f(x) = x^2 + 2$, then if $x = 0, 1, 5$, $y = 2, 3, 27$ respectively.

(2) Periodic function:
A function which repeats itself at a regular interval of x is called periodic.

(3) Integral of Definition:
The range of values of x for y is defined is called integral of definition. For example: If $y = \frac{2}{\sqrt{9-x^2}}$, the function is undefined if $x = 3$ or $x > 3$. Then the integral of definition for this function is $-3 < x < 3$. The function is define for $x = -2, -1, 0, 1, 2$.

(4) Monotonic function:
$f(x_1)$ is monotonic increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
$f(x_1)$ is monotonic decreasing if $f(x-1) > f(x_2)$ whenever $x_1 > x_2$.

(5) Even and Odd function:
A function $f(x)$ is said to be even if $f(x) = f(-x)$.
For example: $f(x) = x^2 + 1$, $f(-x) = (-x)^2 + 1 = x^2 + 1$
A function is said to be odd if $f(-x) = -f(x)$.
For example: $f(x) = x^3, f(-x) = (-x)^3 = -f(x)$

(6) Function:
Given two non-empty sets A and B, if there is a rule, which assigns an element $x \in A$ a unique element $y \in B$, such a rule is called a mapping. A function is a rule for transforming a member of one set A to a unique member of another set B. A function from a set A to as set B is a rule which associates with each member of A a unique member of B. Then $f : A \to B$. A is called the domain of the function and B the codomain. A subset of the co-domain, which ia s collection of all the images of the elements of the domain is called the Range.
Example 1: What is the domain and range of the function $f(x) = x^2$.

Date: April 19, 2012.

Solution: For any real number, its square is uniquely defined. Therefore the domain of f is the set $\mathbb{R}$. The square of any number is never negative and the square root of any positive real number exists. Therefore the range is the set of non-negative real numbers.
Example 2: Find the range and domain of $f(x) = \sqrt{(1-x^2)}$
Solution: The domain is the set $B = \{x \in \mathbb{R} : 1-x^2 \geq 0\}$. Therefore $B = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$
The range is the set of real numbers between 0 and 1, that is $C = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

Graphs of functions

The graph of a function is pictorial representation of the function by use of co-ordinate system. The graph of a function f is the collection of all pairs of numbers $(x, f(x))$ where x is the domain of f. The function $f(x) = x+3$ has a straight line graph (It will be shown in the class). Consider the function $f(x) = x^2$, the graph is the collection of points whose co-ordinate satisfy this equation. The points are $(0,0), (1,1), (2,4), (3,9), (-1,1), (2,4), (-3,9), ...(x, x^2)$. The graph will be shown in the class. The graph of $f(x) = 4, f(x) = x, f(x) = \sqrt{x}, f(x) = x^3, f(x) = sin\ x, f(x) = cos\ x, f(x) = tan\ x$ will be shown during the lecture.

One-to-one functions

Functions for which different inputs always give different output are called one-to-one function (Injective). Thus $f : A \rightarrow B$ is one-to- one , if $f(a) = f(b)$ implies that $a = b$ or $a \neq b$ implies that $f(a) \neq f(b)$.
Note: If one input gives two different outputs, then the mapping is not a function.
For example: If $f(x) = 2x + 1$ and $x = \{3, 4, 5, 6\}$
$f(a) = f(b) \Rightarrow a = b,\ f(3) = 7, f(4) = 9, f(5) = 11, f(6) = 13$

Onto function

These are functions whose range is equal to the codomain (surjective) while the mapping f is bijective if it is both injective and surjective.

Composite Functions

Suppose $f : A \rightarrow B$ and $g :\rightarrow C$ are two functions. Then $g \circ f = A \rightarrow C$ where $g \circ f = g(f(x))$ is the composite function.
For example: If $f : x \rightarrow x^2+2$ and $g : y \rightarrow \sqrt{y+5}$. Find $f(2), f(g(20)), f(g(4)), g(f(4))$.
Solution: $f(2) = 6$
$g(20) = 5$ and $f(g(20)) = 27$
$g(4) = 3$ and $f(g(4)) = 11$
$f(4) = 18$ and $g(f(4)) = \sqrt{22}$

The inverse of a function

Let $f : A \rightarrow B$.The inverse of f, if it exists is the function $y : B \rightarrow A$ such that for all $a \in A$ and all $b \in B$, if $f(a) = b$, then $g(b) = a$ (invertible

function).
Example: If $f : x \to \frac{x+1}{x+2}, g : y \to 3y + 2$. Determine the function $f^{-1}, g^{-1}, f^{-1}(g(1)), f^{-1}(g^{-1}(2)), g^{-1}(f^{-1}(2))$.
Solution: $f : x \to \frac{x+1}{x+2}$
Let $p = \frac{x+1}{x+2}$

$$p(x+2) = x + 1$$
$$px - x = 1 - 2p$$
$$x(p-1) = 1 - 2p$$
$$x = \frac{1-2p}{p-1}$$

Therefore

$$f^{-1} : x \to \frac{1-2x}{x-1}$$

For g^{-1}:

$$g :\to 3y + 2$$
$$Let \quad q = 3y + 2$$
$$y = \frac{q-2}{3}$$
$$g^{-1} : y \to \frac{y-2}{3}$$
$$g(1) = 5, \quad f^{-1}(g(1)) = \frac{-9}{4}$$
$$f^{-1}(g^{-1}(2)) = -1 \quad since \quad g^{-1}(2) = 0$$
$$g^{-1}(f^{-1}(2)) = \frac{-5}{3}$$

Limits

Denote by $lim_{x\to x_0^+} f(x)$ the right hand limit of $f(x)$, that is the value which the function $f(x)$ approaches as x approaches x_0 from the right. Also $lim_{x\to x_0^-} f(x)$ denotes the left hand limit of $f(x)$ as x approaches x_0 from the left. Then $\lim_{x\to x_0} f(x)$ is the limit of $f(x)$ as x approaches x_0 from both left and the right.

Definition of Limits

$\lim_{x\to x_0} f(x) = L$ exists if the following conditions are satisfied.

(1) $f(x)$ is defined in an open interval containing x_0 but not necessarily at x_0.
(2) $lim_{x\to x_0^+} f(x)$ and $lim_{x\to x_0^-} f(x)$ exists, and
(3) $lim_{x\to x_0^+} f(x) = lim_{x\to x_0^-} f(x) = L$

Some limits theorem

If $\lim_{x\to x_0} f(x)$ and $lim_{x\to x_0} g(x)$ exist, then

(1) $\lim_{x\to x_0} c.f(x) = c. \lim_{x\to x_0} f(x)$, for any $c \in \mathbb{R}$.
(2) $\lim_{x\to x_0}[f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$
(3) $\lim_{x\to x_0}[f(x).g(x)] = [\lim_{x\to x_0} f(x)].[\lim_{x\to x_0} g(x)]$

(4) $\lim_{x \to x_0}[f(x)]^n = [\lim_{x \to x_0} f(x)]^n$
(5) $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{\lim_{x \to x_0} f(x)}$, if $\lim_{x \to x_0} f(x) > 0$
(6) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$, If $\lim_{x \to x_0} g(x) \neq 0$
(7) Limits of polynomial and Rational Function:
If $f(x) = p(x) = a_0 + a_1 x + ... + a_n x^n, x \in R$ is a polynomial, then $\lim_{x \to x_0} p(x) = p(x_0)$ for any $x_0 \in \mathbb{R}$.
(8) If $f(x) = p(x)$ and $g(x) = q(x)$ are polynomials and $q(x_0) \neq 0$, then $\lim_{x \to x_0} \frac{p(x)}{q(x)} = \frac{p(x_0)}{q(x_0)}$.
(9) Infinite Limits:
$\lim_{x \to x_0} \frac{1}{x^{2r}} = +\infty$ for any positive integer r.
(10) Limits at Infinity:
$lim_{x \to +\infty} \frac{1}{x^r} = 0$, for any $r \in \mathbb{R}, r > 0$ $lim_{x \to -\infty} \frac{1}{x^r} = 0$, for any $r \in \mathbb{R}, r > 0$
(11) If $p(x)$ and $q(x)$ are polynomials, such that $deg\ p(x) < deg\ q(x)$, then $lim_{x \to +\infty} \frac{p(x)}{q(x)} = 0$.
(12) If $p(x)$ and $q(x)$ are polynomials, such that $deg\ p(x) = deg\ q(x)$, then $lim_{x \to +\infty} \frac{p(x)}{q(x)} = L$, a finite number.
(13) If $p(x)$ and $q(x)$ are polynomials, such that $ded\ p(x) > deg\ q(x)$, then $lim_{x \to +\infty} \frac{p(x)}{q(x)} = \pm\infty$

Example 1: Find the limits if it exists
(a) $lim_{x \to 1} \frac{x^2-1}{x-1}$
(b) $lim x \to 0|x|$
Solution:
(a) If $f(x) = \frac{x^2-1}{x-1}$, then $f(1)$ does not exist. However $f(x) = \frac{x^2-1}{x-1} = x+1$ if $x \neq 1$
Hence $lim_{x \to 1} \frac{x^2-1}{x-1} = lim_{x \to 1}(x+1) = 2$
(b) $|x| = \{^{x, \quad if \quad x \geq 0}_{-x, \quad if \quad x<0}$
$lim_{x \to 0^+}|x| = 0 = lim_{x \to 0^-}|x|$
$\Rightarrow lim_{x \to 0}|x| = 0$

Example 2: Determine the limit
(a) $lim_{x \to 3}(x^3 - 2x + 6)$
(b) $lim_{x \to -1}(x^2 - 3)^{10}$
(c0 $lim_{x \to 2}(\frac{x^3-3x+6}{-x^2+15})$
Solution:
(a) $lim_{x \to 3}(x^3 - 2x + 6) = 3^3 - 2(3) + 6 = 27$
(b) $lim_{x \to -1}(x^2 - 3)^{10} = 1024$
(c) $lim_{x \to 2}(\frac{x^3-3x+6}{-x^2+15}) = \frac{8}{11}$
Example 3: Obtain the limit
(a) $lim_{x \to +\infty} \frac{x^3+3x+6}{x^5+2x^2+9}$
(b) $lim_{x \to +\infty} \frac{2x^2-2x+3}{x^2+4x+4}$
Solution:

(a) $lim_{x\to+\infty}\frac{x^3+3x+6}{x^5+2x^2+9}$
Divide through by the highest power of x
$= lim_{x\to+\infty}\frac{\frac{1}{x^2}+\frac{3}{x^4}+\frac{6}{x^5}}{1+\frac{2}{x^3}+\frac{9}{x^5}} = 0$
(b) $lim_{x\to+\infty}\frac{2x^2-2x+3}{x^2+4x+4} = 2$
Example4: Find the limit of the function $\frac{x^2-4}{x-2}$ as $x \to 2$ by Le'hospital's rule.
Solution:
Differentiate both the numerator and denominator with respect to x. Then we have $2x$. $lim_{x\to 2}2x = 4$

Continuous function

A function $f(x)$ is said to be continuous at a point x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$, that is: A function $y = f(x)$ is continuous at a point x_0 if

(1) it is defined in a neighborhood of that point x_0
(2) the limit of the function as x tends to x_0 exist.
(3) this limit is equal to the value of the function at the point $x = x_0$.

Example: Check if the following functions are continuous at the given points:
(a) $f(x) = \frac{x}{x^2-2} \quad at \quad x = 1$
(b) $f(x) = \frac{1}{x-1} \quad at \quad x = 1$
Solution:
(a) $f(x) = \frac{x}{x^2-2} \quad at \quad x = 1$

(1) $f(1) = \frac{1}{1-2} = -1$ hence $f(x)$ is defined at $x = 1$
(2) $lim_{x\to 1}\frac{x}{x^2-2} = -1$; the limits exists.
(3) $lim_{x\to 1}\frac{x}{x^2-2} = f(1)$

Therefore the conditions are satisfied, the function $f(x)$ is continuous at $x = 1$.
(b) $f(x) = \frac{1}{x-1} \quad at \quad x = 1$

(1) $f(1) = \frac{1}{1-1} = \infty; f(x)$ is not defined at $x = 1$.
(2) $lim_{x\to 1}f(x)$ does not exist at the point $x = 1$.

Since one of the conditions have been violated then $f(x)$ is not continuous at the point $x = 1$.

Limits and continuity of functions of several variables

The function $f(x, y)$ said to tend to limit L as $x \to x_0$ and $y \to y_0$ written as

$$lim_{x\to x_0, y\to y_0}f(x) = L$$

If the limit L is independent of the path followed by the point (x, y) as $x \to x_0$ and $y \to y_0$.
Example: If $f(x, y) = \frac{3x+1}{x^2+y+1}$, find $lim_{x\to 1, y\to 2}f(x)$.
Solution: $lim_{x\to 1, y\to 2}f(x) = \frac{3(1)+1}{1^2+2+1} = 1$.

Also, the function $f(x, y)$ is said to be continuous at the point (x_0, y_0) if $lim_{x \to x_0, y \to y_0} f(x) = L$ exists and $f(x_0, y_0) = L$.

Discontinuous functions

If a function $f(x)$ is not continuous at a point x_0 then it is said to be discontinuous at the point x_0 and the point x_0 is called a point of discontinuity of the function.

There are basically two major types of discontinuities.

(1) Removable discontinuity: If $\lim_{x \to x_0} f(x)$ exists and is unequal to $f(x_0)$ then x_0 is said to be a point of removable discontinuity of $f(x)$. If that happens, by redefining the function $f(x)$ in a way such that $f(x_0) = \lim_{x \to x_0} f(x)$, then $f(x)$ can be made to be continuous at $x = x_0$.
Example: Show that the function $f(x) = \frac{x^2-2}{x-2}$ has a removable discontinuity at the point $x = 2$.
Solution: Since $f(x)$ is not defined at $x = 2$. Apply Le'Hospital rule to have $lim_{x \to 2} f(x) = 4$
Redefine the function as $f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$
then $f(2) = 4 \Rightarrow f(2) = lim_{x \to 2} f(x) = 4$
Thus the function is now continuous at $x = 2$.

(2) Non-Removable Discontinuity: If the right and left hand limits exists but unequal, that is $lim_{x \to x_0^+} f(x) \neq lim_{x \to x_0^-} f(x)$ or either the $lim_{x \to x_0^+} f(x)$ or $lim_{x \to x_0^-} f(x)$ does not exist then such function $f(x)$ is said to have non-removable discontinuity at $x = x_0$
Example: The function $f(x) = sin\, \frac{1}{x}$ is continuous for $x \neq 0$. The function has non-removable discontinuity at $x = 0$. Both right and left hand limits does not exist.