<u>MCE 306: Fluid Mechanics III (2 Units)</u>

Course Synopsis

Kinematics of fluid: Eularian and Lagrangian descriptions. The stream function. Sources, sinks and doublets. Streamline bodies including aerofoils and hydrofoils. Circulation, vorticity and vortices. Irrotational flow and velocity potential. Laminar internal flows, flow through straight channels and **covette** flow. Very slow motion and lubrication. Turbulent internal flow. Non-circular pipe flow. Piping design. Elements of compressible flow.

Textbooks

1 Kundu, P. K., Cohen, I. M. (2002). Fluid Mechanics, Second Edition, Academic Press, San Diego, USA.

2 Kothandaraman, C. P., Rudramoorthy, R. (2007). Fluid Mechanics and Machinery, New Age International (P) Limited, New Delhi, Indian.

3 Duncan, W. J., Thom, A. S., Young, A. D. (1962).An Elementary Treatise on the Mechanics of Fluids, Edward Arnold (Publishers) Ltd., London, UK.

4 Shames, I. H. (1982). Mechanics of Fluids, McGraw-Hill International Book Company, Auckland, Australia, Second Edition.

5 Douglas, J. F., Gasiorek, J. M., Swaffield, J. A. (1983). Fluid Mechanics, Pitman Books Limited, Massachusetts, UK.

6 Streeter, V. L., Wylie, E. B. (1975). Fluid Mechanics, McGraw-Hill Kogakusha Ltd., Tokyo, Japan, Sixth Edition.

7 Massey, B. S. (1980). Mechanics of Fluids, Nostrand Reinhold Company Ltd., Berkshire, England, Fourth Edition.

LECTURE NOTE ON MCE 306: FLUID MECHANICS III

CHAPTER 1: KINEMATICS OF FLUIDS

1.0 MECHANICS

Mechanics is a branch of physical sciences concerned with the state of rest or motion of bodies that are subjected to the action of force. It is subdivided into three branches: rigid-body or classical mechanics, deformable-body mechanics, and fluid mechanics.

Rigid-body mechanics is generally divided into two areas: statics and dynamics. Statics deals with the equilibrium of bodies, that is, those which are either at rest or move with a constant velocity.

Dynamics is concerned with the accelerated motion of bodies. The subject of dynamics is usually divided into two parts: (1) kinematics is concerned with the geometrical aspects of motion, and (2) kinetics is concerned with the analysis of the forces causing the motion.

1.1 KINEMATICS (HYDRODYNAMICS)

Kinematics or Hydrodynamics

Kinematics is the branch of mechanics that deals with quantities involving space and time only. It treats variables such as displacement, velocity, acceleration, deformation, and rotation of fluid elements without referring to the forces responsible for such a motion.

In fluid mechanics, the study of the velocity of various particles on the flow and the instantaneous flow pattern of the flow field is called flow kinematics or hydrodynamics. A thorough study of the kinematics of fluids is a necessary preliminary to the study of the dynamics of fluids. Kinematics investigations carry us very far into the general theory of fluid motion. The kinematics of fluids presents problems of much greater complexity than does the kinematics of rigid bodies and requires quite different theoretical methods for its treatments. In this class we shall study the followings:

- Description of a Fluid Field
- Substantive or Material Derivative
- Streamlines, Trajectories and Streaklines

It should be noted that the fluid is taken as a continuum. This means that there is no gap in the fluid, i.e. we are now replacing the fluid with continuous molecules. There are cases, however, where this does not apply, i.e. at extremely low pressure.

Continuum assumes the fluid to be a continuous material even though we know that matter consists of myriads of molecules in constant motion and collision.

1.1 Description of a Fluid Field – Coordinate Systems

There are basically two methods to describe flow trajectory: the Eulerian and the Lagrangian method.

Lagrangian Method

In the Lagrangian approach, one essentially follows the history of individual fluid particles. The two independent variables are taken as time and a label for fluid particles. The label can be conveniently taken as the position vector x_o of the particle at some reference time $t = 0$. Any flow variable F is expressed as $F(x_{o,}\, t)$, i.e.

$$x = x(x_o, t)$$

$$u = u(x_o, t) = \frac{dx}{dt}$$

$$a = a(x_o, t) = \frac{du}{dt} = \frac{d^2x}{dt^2} \qquad (1.1)$$

$x(x_o, t)$, $u(x_o, t)$ and $a(x_o, t)$ represents the location, velocity and acceleration at time t of a particle whose position was x_o at $t = 0$.

In 3-dimensional flow, the position vector is defined as:

$$x = x(x_o,\ y_o, z_o, t)\ ,\ y = y(x_o,\ y_o, z_o, t)\ ,\ z = z(x_o,\ y_o, z_o, t)$$

The velocity as:

$$u(x_o,\ y_o, z_o, t) = \frac{dx(x_o,\ y_o, z_o, t)}{dt}$$

$$v(x_o,\ y_o, z_o, t) = \frac{dy(x_o,\ y_o, z_o, t)}{dt}$$

$$w(x_o,\ y_o, z_o, t) = \frac{dy(x_o,\ y_o, z_o, t)}{dt} \qquad (1.2)$$

And the acceleration as:

$$a_x(x_o,\ y_o, z_o, t) = \frac{du(x_o,\ y_o, z_o, t)}{dt} = \frac{d^2x(x_o,\ y_o, z_o, t)}{dt^2}$$

$$a_y(x_o,\ y_o, z_o, t) = \frac{dv(x_o,\ y_o, z_o, t)}{dt} = \frac{d^2y(x_o,\ y_o, z_o, t)}{dt^2}$$

$$a_z(x_o,\ y_o, z_o, t) = \frac{dw(x_o,\ y_o, z_o, t)}{dt} = \frac{d^2z(x_o,\ y_o, z_o, t)}{dt^2} \qquad (1.3)$$

Eulerian Description

In this approach, one concentrates on what happens at a spatial point, *x*, so that the independent variables are taken as *x* and *t*. That is a flow variable is written as $F(x, t)$, i.e. the velocity *u*, *v* and *w* in *x*-, *y*- and *z*-direction, and acceleration *a*

$$u = u(x, y, z,\ t)$$
$$v = v(x, y, z,\ t)$$
$$w = w(x, y, z,\ t)$$
$$a = a(x, y, z,\ t) \qquad (1.4)$$

It will soon be clear that in Eulerian method the partial or total derivative gives only local rate of change at a point *x* and is not the total rate of change seen by a fluid particle.

The relationship between the Lagrangian and the Eulerian description follows from the fact that the velocity at position *x* and time *t* must be equal to the velocity of the fluid particle which is at this position and at this particular time, i.e.

$$\frac{dx(x_o, t)}{dt} = u(x, t) \qquad (1.5)$$

From a practical point of view the Eulerian description is the easier one to use. The Lagrangian description, however, has advantages mainly from a theoretical point of view over the Eulerian method, e.g. in the formulation of fluid motion governing equation, in the study of dispersion of contaminants, etc.

1.2 Material Substantive or Total Derivative

As early mentioned the Eulerian description is most commonly used in practice when we want to describe a fluid motion. However, we need to express Lagrangian properties of the flow, i.e. properties of individual fluid elements, in an Eulerian frame of reference. For instance we may ask ourselves what is the acceleration experienced by a fluid element expressed in a Eulerian system.

Let us first consider a function $G(x, y, z, t)$ which is a continuously differentiable function of the coordinates (x, y, z, t). This means that all partial derivatives of G exist. Let us interpret G as the property of a fluid element, which is at the position $\boldsymbol{P}(x, y, z, t)$. Examples of relevant properties are for instance: density, temperature or pressure. We now want to express the change of this property as a function of time when the fluid element moves along its trajectory. This is called the **material derivative** of G and it is expressed by the following notation: DG/Dt. This is expressed as:

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} + \frac{\partial G}{\partial z}\frac{dz}{dt} \qquad (1.6)$$

The vector $(dx/dt, dy/dt, dz/dt)$ describes an arbitrary path through three-dimensional space as a function of time. Thus DG/Dt can be expressed as:

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + u\frac{\partial G}{\partial x} + v\frac{\partial G}{\partial y} + w\frac{\partial G}{\partial z} \qquad (1.7)$$

It then follows that the material derivative in vector and Cartesian tensor notation can be written as

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + (u \bullet grad)G \qquad (1.8)$$

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + u_i\frac{\partial G}{\partial x_i} \qquad (1.9)$$

In equations (1.7), (1.8) and (1.9) $\partial G/\partial t$ is usually denoted as the local derivative or local part of the total derivative because it describes the change of G as a function of time at a fixed position in space (note the partial derivative). All the other terms on the right-hand side of equations (1.6) and (1.7) is usually denoted as the advective derivative or advection or sometime also convective derivative or convection. This advective derivative or rather the advective part of the total derivative gives the change of G as a function of time resulting from the fact that the fluid element moves in a non-homogeneous scalar field $G(x, y, z)$.

For a property G of a fluid element which does not change along its trajectory, we find thus immediately the equation

$$\frac{DG}{Dt} = 0 \qquad (1.10)$$

Any property, which satisfies equation (1.10) is called a material property. An example is the interface between two immiscible fluids, which moves with the flow at the position of the interface and is thus a material property.

Above we have expressed the material derivative for a scalar property *G*. However, the material derivative can be also extended to a vector property. Let us take as an example the flow velocity G ≡ G(u). The material derivative of the velocity at position and time (x, y, z, t) can be interpreted as the acceleration of the fluid element, which is at time t on the position x where the material derivative is taken. When the acceleration a_x in x-direction is calculated in a Cartesian frame of reference then there is no problem and we can basically extend equation (1.6) to the velocity leading to

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (1.11)$$

In Cartesian tensor notation, equation (1.11) can be written as

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} \qquad (1.12)$$

Note that repeated indices have to be summed over all coordinates, which is called the Einstein summation convention.

1.3 Conservation Laws of Physics

The governing equations of fluid flow represent mathematical statements of conservation of physics, i.e.

- The mass of a fluid is conserved
- The rate of change of momentum equals the sum of the forces on a fluid particle (Newton's second law)
- The rate of change of energy in equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle (first law or thermodynamics)

Basic Scientific Laws Used in the Analysis of Fluid Flow

- Law of conservation of mass
- Newton's laws of motion
- Law of conservation of energy
- Thermodynamic law
- Equation of state

The fluid may be regarded as continuum. We describe the behaviour of the fluid in terms of macroscopic properties such as velocity, density, pressure and temperature, and their space and time derivatives.

Mass Conservation in Three Dimensions

The first step in the derivation of mass conservation equation is to write down a mass balance for the fluid element.

Rate of increase of mass in fluid element = Net rate of flow of mass into fluid element (1.13)

Consider mass flow rate in and out of fluid element shown in Fig. 1.1.

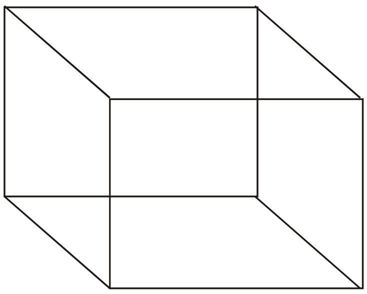


Figure 1.1: Mass flow in and out fluid element.

Using the differential expression approach, the rate of increase of mass in the fluid element is

$$\frac{\partial}{\partial t}(\rho\delta x\delta y\delta z) = \frac{\partial\rho}{\partial t}(\delta x\delta y\delta z) \quad (1.14)$$

Mass flow rate across a face of the element:

x-direction: $\left[\rho u-\left(\rho u+\frac{\partial(\rho u)}{\partial x}\delta x\right)\right]\delta y\delta z=-\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z$ (1.15)

y-direction: $\left[\rho v-\left(\rho v+\frac{\partial(\rho v)}{\partial y}\delta y\right)\right]\delta x\delta z=-\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z$ (1.16)

y-direction: $\left[\rho w-\left(\rho w+\frac{\partial(\rho w)}{\partial z}\delta z\right)\right]\delta x\delta y=-\frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$ (1.17)

Net rate of flow of mass into the fluid element is:

$$=-\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z-\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z-\frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z \tag{1.18}$$

Substituting equations (1.14) and (1.18) in equation (1.13) gives:

$$\frac{\partial\rho}{\partial t}(\delta x\delta y\delta z)=-\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z-\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z-\frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

which simplifies into

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \tag{1.19}$$

In vector form:

$$\frac{d\rho}{dt}+\nabla\cdot(\rho v)=0 \tag{1.20}$$

Equation (1.19) or (1.20) is referred to as the continuity equation. It is a general expression for the conservation of mass in differential form.

In equation (1.20),

$$\nabla=\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z \tag{1.21}$$

and the velocity vector,

$$\vec{v}=u\vec{i}+v\vec{j}+w\vec{k} \tag{1.22}$$

If the flow field has a sink or a source of strength $\dot{m}'''$, equation (1.19) or (1.20) will become:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\dot{m}''' \tag{1.23}$$

and in vector form,

$$\frac{d\rho}{dt} + \nabla \cdot (\rho v) = \dot{m}''' \tag{1.24}$$

Continuity Equation – Control Volume Approach

Continuity equation can be also derived using the integral approach as follows:

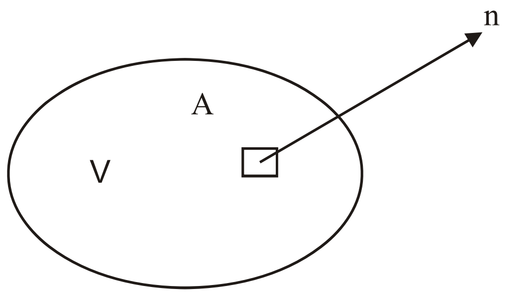


Figure 1.2: Illustration of the geometry of control volume V.

Consider the control volume shown above in which fluid flows in and out.

The mass of the fluid inside our control volume is $\iiint_V \rho dV$.

For a control volume fixed is space, the rate of change of mass inside of our control volume is

$$\frac{d}{dt}\iiint_V \rho dV = \iiint_V \frac{d\rho}{dt}\, dV \tag{1.25}$$

The rate at which mass enters the control volume through its surface is

$$\iint_S \rho v \cdot n dS \tag{1.26}$$

where $\rho v \,.\, n\mathrm{d}S$ is the mass rate of flow out of the small area dS. The quantity $v.n$ is the normal component of the velocity to the surface.

The net rate of change of mass inside and entering the control volume is then found by adding together equations (1.25) and (1.26).

$$\iiint_V \frac{d\rho}{dt}\, dV + \iint_S \rho v \cdot n dS = 0 \tag{1.27}$$

Transforming the surface integral to volume integral using Gauss divergence theorem (Green's theorem):

$$\iint_S \rho v \cdot n dS = \iiint_V \nabla \cdot (\rho v) dV \tag{1.28}$$

Equation (1.28) becomes:

$$\iiint_V \frac{d\rho}{dt}\, dV + \iiint_V \nabla\cdot(\rho v) dV = 0 \qquad (1.29)$$

$$\iiint_V \left[\frac{d\rho}{dt} + \nabla\cdot(\rho v)\right] dV \;= 0$$

$$\therefore \frac{d\rho}{dt} + \nabla\cdot(\rho v) \;= 0 \qquad (1.30)$$

Equation (1.30) is exactly the same as equations (1.19) or (1.20).

Special Forms of the Continuity Equation

(i) Steady flow with source or sink

$$\nabla\cdot(\rho v) \;= \dot{m}'''$$

(ii) Steady flow without source or sink

$$\nabla\cdot(\rho \boldsymbol{v}) \;= 0$$

(iii) Unsteady flow without source/sink

$$\frac{d\rho}{dt} + \nabla\cdot(\rho \boldsymbol{v}) \;= 0$$

(iv) Incompressible flow with source/sink

$$\nabla\cdot\boldsymbol{u} \;= \frac{\dot{m}'''}{\rho}$$

(v) Incompressible flow without source or sink

$$\nabla\cdot\boldsymbol{u} \;=$$

Momentum Equation in Three Dimensions

Newton's second law states that the rate of change of momentum of a fluid particle equals the sum of the forces on the particle.

i. e. Rate of Increase of Momentum = Sum of Forces on Particles

The rate of increase of x-, y- and z- momentum per unit volume of a fluid particle are given by $\rho\frac{Du}{Dt}, \rho\frac{Dv}{Dt}, and\ \rho\frac{Dw}{Dt}$ respectively.

We distinguish two types of forces in fluid particles: surface forces and body forces.

Surface forces - As the name indicates these forces act upon the surface of the fluid particle or upon the surface of the considered fluid domain, pressure forces, viscous forces

Body forces – This type of forces acts upon the whole material volume at a distance, without contact with the body, e.g. gravity force, centrifugal force, coriolis force, electromagnetic force

Line forces or surface tension – These are other type of forces that are considered in fluid flow. At interfaces between two substances, the inter-molecular forces at both sides differ, appearing to be an additional force. At the macroscopic level, the interfacial forces can be modeled by the

$$Surface\ tension\ \sigma = \frac{force}{length} \tag{1.31}$$

which causes a force tangent to the interface and orthogonal to any line through the interface, of modulus

$$dF_l = \sigma\, dl \tag{1.32}$$

The surface tension depends on the pair of substances that form the interface and on the temperature. When the surface tension is positive, the molecules of each phase tend to be repelled back to their own phase. This is the case, for instance, of two inmiscible liquids. When the surface tension is negative, the molecules of both phases tend to mix, like two miscible liquids. In the case of a liquid/gas interface, the surface tension tends to maintain the interface (or free surface) straight. An important situation appears when three substances meet forming three interfaces, for instance, at a wall/liquid/gas interface. In this case, the line, which is the intersection of the three interfaces, is called the contact line. The angle that two interfaces form at the contact line is called the contact angle and depends on the surface tension of all interfaces. Therefore, the contact angle depends solely on the three substances and the temperature.

Finally, to derive the momentum equation, we equate the flow inertial force to the summation of all the forces in the flow, i.e.

$$\rho \frac{Du}{Dt} = \sum F_i \tag{1.33}$$

1.4 Flow Vusualization

Pathlines, Streaklines, Timelines, Streamlines, Streamtubes and Stream Function

Just like the topography of a region is visualized using the contour map, flow can be visualized using the velocity at all points at a given time or the velocity of a given particle at different time.

Pathline is the trace of the path of a single particle over a period of time. Pathline shows the direction of the velocity of a particle at successive instants of time. It is best described using the Lagrangian description. The equation of a path line is given by:

$$dt = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad (1.34)$$

the integration being performed with x_o, y_o, z_o held fixed.

Streaklines provide an instantaneous picture of the particles, which have passed through a given fixed point. A streakline results when we release smoke or dye at a fixed point in the flow field. For a stationary flow the streakline, streamline and trajectory are identical.

Timeline: If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line at that instant. This line is called timeline.

Steamlines are series of curves drawn tangent to the mean velocity vectors of a number of particles in a flow. Since streamlines are tangent to the velocity vector at every point in the flow field, there can be no flow across a streamline. The requirement of tangency means that the streamlines are given by the equation

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad (1.35)$$

In a steady flow, pathlines and streamlines will be identical, streaklines and will also coincide with stream lines. This implies that in a steady flow, streaklines, streamlines, pathlines, and trajectory are identical.

A **stream surface (or stream sheet)** is a collection of adjacent streamlines, providing a surface through which there is no flow.

A **streamtube** is a tube made up of adjoining stream lines. A bundle of neighbouring streamlines may be imagined to form a passage through which the fluid flow such a passage is called a stream tube. The consequence is that there is no transport through the side walls, because at every point the velocity is parallel to the local velocity vector, or in other words transport, e.g. of mass, through each cross section of the tube must be the same.

Example 1: (Streamline). Calculate the streamlines for the unsteady, two-dimensional flow field given by, $u = 2x(t + 1)$; $v = 2y(t - 1)$

Particularize for the case in which the streamline passes through the point (x_0, y_0) at all times.

Solution:

Applying equation (1.31)

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

It follows that: $\dfrac{dx}{2x(t+1)} = \dfrac{dy}{2y(t-1)}$

Integrating: $(t+1) ln\ y = (t-1) ln\ x + ln\ C$

Thus $y^{t+1} = Cx^{t-1}$

To determine the integration constant C, the conditions of the particular case are imposed for all t,

$$y_o^{t+1} = Cx_o^{t-1}$$

And so $C = \dfrac{y_o^{t+1}}{x_o^{t-1}}$

Finally, substituting the value of C:

$$\frac{y}{y_o} = \left(\frac{x}{x_o}\right)^{\frac{t-1}{t+1}}$$

Example 2: (Trajectory). For the flow field of the above example, determine the trajectory of the fluid particle that passes through the point (x_0, y_0), at $t = 0$.

Solution: Integrating the equation of motion:

$$dx = 2x(t+1)dt$$

$$dy = 2y(t-1)dt$$

Yields:

$$ln\ x = (t+1)^2 + ln\ C_1$$

$$ln\ y = (t-1)^2 + ln\ C_2$$

Thus $$x = C_1 e^{(t+1)^2}$$

$$y = C_2 e^{(t-1)^2}$$

To determine the constants of the integration C_1, C_2, the conditions of the problem are imposed:

$$x_o = C_1 e^{(0+1)^2}$$

$$y_o = C_2 e^{(0-1)^2}$$

which implies $C_1 = x_o/e$

$$C_2 = y_o/e$$

Finally, the trajectory is given in parametric form through the combination of

$$\frac{x}{x_o} = e^{(t+1)^2 - 1}$$

$$\frac{y}{y_o} = e^{(t-1)^2 - 1}$$

This is a valid curve in two dimensions. Sometimes it is possible to eliminate *t* and write the same curve in explicit form, that is, as *y*(*x*). Getting t from the first equation,

$$t = \sqrt{ln\ \frac{x}{x_o} + 1} - 1$$

and substituting in the second one,

$$\frac{y}{y_o} = e^{\left(\sqrt{ln\ x/x_o + 1} - 2\right) - 1}$$

which is the equation of the trajectory in explicit form.

1.5 Definition of Basic Types of Flow

Flow of Ideal/Inviscid and Real Fluids

Ideal fluid is non-viscous and incompressible. Shear force between the boundary surface and fluid or between the fluid layers is absent and only pressure forces and body forces are controlling.

Real fluids have viscosity and surface shear forces are involved during flow. However the flow after a short distance from the surface is not affected by the viscous effects and approximates to ideal fluid flow. The results of ideal fluid flow analysis are found applicable in the study of flow of real fluids when viscosity values are small.

Steady and Unsteady Flow

In order to study the flow pattern it is necessary to classify the various types of flow. The classification will depend upon the constancy or variability of the velocity with time. In the next three sections, these are described. In steady flow the property values at a location in the flow are constant and the values do not vary with time. The velocity or pressure at a point remains constant with time. These can be expressed as $V = V(x, y, z)$, $P = P(x, y, z)$ etc.

In steady flow a picture of the flow field recorded at different times will be identical. In the case of unsteady flow, the properties vary with time or $V = V(x, y, z, t)$, $P = P(x, y, z, t)$ where t is time. In unsteady flow the appearance of the flow field will vary with time and will be constantly changing. In turbulent flow the velocity at any point fluctuates around a mean value, but the mean value at a point over a period of time is constant. For practical purposes turbulent flow is considered as steady flow as long as the mean values of properties do not vary with time.

Compressible and Incompressible Flow

If the density of the flowing fluid is the same all over the flow field at all times, then such flow is called incompressible flow. Flow of liquids can be considered as incompressible even if the density varies a little due to temperature difference between locations. Low velocity flow of gases with small changes in pressure and temperature can also be considered as incompressible flow. Flows through fans and blowers are considered incompressible as long as the density variation is below 5%. If the density varies with location, the flow is called compressible flow. In this chapter the study is mainly on incompressible flow.

Laminar and Turbulent Flow

If the flow is smooth and if the layers in the flow do not mix macroscopically then the flow is called laminar flow. For example a dye injected at a point in laminar flow will travel along a continuous smooth line without generally mixing with the main body of the fluid. Momentum, heat and mass transfer between layers will be at molecular level of pure diffusion. In laminar flow layers will glide over each other without mixing.

In turbulent flow fluid layers mix macroscopically and the velocity/temperature/mass concentration at any point is found to vary with reference to a mean value over a time period. For example $u = u + u'$ where u is the velocity at an instant at a location and u is the average velocity over a period of time at that location and u' is the fluctuating component. This causes higher rate of momentum/heat/mass transfer. A dye injected into such a flow will not flow along a smooth line but will mix with the main stream within a short distance. The difference between the flows can be distinguished by observing the smoke coming out of an incense stick. The smoke in still air will be found to rise along a vertical line without mixing. This is the laminar region. At a distance which will depend on flow conditions the smoke will be found to mix with the air as the flow becomes turbulent. Laminar flow will prevail when viscous forces are larger than inertia forces. Turbulence will begin where inertia forces begin to increase and become higher than viscous forces.

Concepts of Uniform Flow, Reversible Flow and Three Dimensional Flow

If the velocity value at all points in a flow field is the same, then the flow is defined as uniform flow. The velocity in the flow is independent of location. Certain flows may be approximated as uniform flow for the purpose of analysis, though ideally the flow may not be uniform. If there are no pressure or head losses in the fluid due to frictional forces to be overcome by loss of kinetic energy (being converted to heat), the flow becomes reversible. The fluid can be restored to its original condition without additional work input. For a flow to be reversible, no surface or fluid friction should exist. The flow in a venturi (at low velocities) can be considered as reversible and the pressures upstream and downstream of the venturi will be the same in such a case. The flow becomes irreversible if there are pressure or head losses. If the components of the velocity in a flow field exist only in one direction it is called one dimensional flow and $V = V(x)$. Denoting the velocity components in x, y and z directions as u, v and w, in one dimensional flow two of the components of velocity will be zero. In two dimensional flow one of the components will be zero or $V = V(x, y)$. In three dimensional flow all the three components will exist and $V = V(x, y, z)$. This describes the general steady flow situation. Depending on the relative values of u, v and w approximations can be made in the analysis. In unsteady flow $V = V(x, y, z, t)$.

Problem set 1

1 Write the continuity equation in cylindrical and spherical coordinate systems.

2 A vertical cylindrical tank closed at the bottom is partially filled with an incompressible liquid. A cylindrical rod of diameter d_i (less than the tank diameter, d_o) is lowered into the liquid at a velocity V. Determine the average velocity of the fluid escaping between the rod and the tank walls

(a) relative to the bottom of the tank

(b) relative to the advancing rod.

3 Determine if the following flows of an incompressible fluid satisfy the continuity equation

(a) $$u = \left[\frac{1}{x^2 + y^2} - \frac{2x^2}{\left(x^2 + y^2\right)^2}\right] V_o r_o^2$$

$$v = -\frac{2xy}{\left(x^2 + y^2\right)^2} V_o r_o^2$$

Where V_o is a reference velocity and r_o is a reference length. Both are constants.

(b) $$u = -\frac{2xyz}{\left(x^2 + y^2\right)^2} V_o r_o$$

$$v = -\frac{\left(x^2 - y^2\right)z}{\left(x^2 + y^2\right)^2} V_o r_o$$

$$w = \frac{y}{x^2 + y^2} V_o r_o$$

4 For the flow of an incompressible fluid the velocity component in the x-direction

$$u = ax^2 + by$$

and the velocity component in the z-direction is zero. Find the velocity components v in the y-direction. In evaluating the arbitrary functions which might appear in the integration, assume that v= 0 at y = 0.

5 Obtain Euler's equation in plane, polar coordinates

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + f_r$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} = -\frac{1}{\rho}\frac{\partial p}{r\partial \theta} + f_\theta$$

By considering the forces on a small element bounded by the lines corresponding to r, r + dr, and θ, θ + dθ.

6 Check whether the following incompressible flow fluid

(a) satisfies the continuity equation

(b) is rotational or irrotational.

$$\vec{v} = 2x\vec{i} - 2y\vec{j} + 16\vec{k}$$

7 Given a velocity field

$$\vec{v} = \left(16x^2 + y\right)\vec{i} + 10\vec{j} + yz^2\vec{k}$$

Evaluate the (i) angular velocity of rotation of a fluid element at the position (ii) the vorticity of the fluid $\vec{r} = 16\vec{i} + 3\vec{j} + 2\vec{k}$

8 Given that the fluid flows from a large reservoir to form the flow field $\vec{v} = 2x\vec{i} - 2y\vec{j} + \left(3t^2 + 16\right)\vec{k}$ of an incompressible fluid of density ρ, what is the difference in static pressure in terms of ρ between points $\vec{r}_1$ and $\vec{r}_2$ when $\vec{r}_1 = 3\vec{i} + 6\vec{j} + 12\vec{k}$ and $\vec{r}_2 = 5\vec{i} - 3\vec{j} + 12\vec{k}$

9 Consider a two-dimensional flow with velocity components $u = -\alpha x + \beta t$ and $v = \alpha y + \gamma t$. Compute the stream-line pattern and the *x*- and *y*-component of the trajectories.

10 Sketch the streamlines for the flow

$u = \alpha x,\ v = -\alpha x,\ w = 0$

where α is a positive constant. Let the concentration of some pollutant in the fluid be

$c\ (x, y, t) = \beta x2ye{-}\alpha t,$

for $y > 0$, where β is a constant. Does the pollutant concentration for any particular fluid element change with time?

11 What is the acceleration of a particle at (3, 0, 2) m at time t = 1 s? if the flow field is given as:

(a) $\vec{v} = \left(6 + 2xy + t^2\right)\vec{i} - \left(xy^2 + 10t\right)\vec{j} + 25\vec{k} \quad (m/s)$. **[a = - 58i – 10j m/s2]**

(b) $\vec{v} = 6x\vec{i} + 6y\vec{j} - 2t\vec{k} \ (m/s)$

12 Determine (i) the equation of the streamlines at $t = 0$ up to an arbitrary constant and (ii) the slope of the streamlines at $t = 0$ s of a flow field with the velocity field $\vec{v} = 6x\vec{i} + 6y\vec{j} - 2t\vec{k}\ (m/s)$. Also sketch the streamlines at $t = 0$.

13 A flow field is given as $\vec{v} = 6x\vec{i} + 6y\vec{j} - 2t\vec{k} \ (m/s)$. What is the velocity at position x = 10 m and y = 6 m when t = 10 s? What is the slope of the streamlines for this flow at t = 0 s? What is the equation of the streamlines at t = 0 s up to an arbitrary constant? Finally, sketch streamlines at t = 0 s.

14 Consider the instationary flow $u = u_0,\ v = kt,\ w = 0,$ where u_0 and k are positive constants. Show that the streamlines are straight lines, and sketch them at two different times. Also show that any fluid particle follows a parabolic path as time proceeds.

15 Water flows through a pipe AB of diameter d_1 = 50 mm, which is in series with a pipe BC of diameter d_2 = 75 mm in which the mean velocity v_2 = 2 m/s. At C the pipe forks and one branch CD is of diameter d_3 such that the mean velocity is 1.5 m/s. The other branch CE is of diameter d_4 = 30 mm and conditions are such that the discharge Q_2 from BC divides so that $Q_4 = \frac{1}{2} Q_3$. Calculate the values of Q_1, v_1, Q_2, Q_3, d_3,Q_4 and v_4. (Douglas et al., 1983, pp. 110).

Figure .

16 The velocity of a fluid varies with time t. Over the period from t = 0 to t = 8 s the velocity components are u = 0 m/s and v = 2 m/s, while from t = 8 s to t = 16 s the components are u = 2 m/s and v = -2 m/s. A dye streak is injected into the flow at a certain point commencing at time t = 0 and the path of a particle of fluid is also traced from that point starting at t = 0. Draw to scale the streakline, pathline of the particle and the streamlines at time t = 12 s. (Douglas et al., 1983, pp. 113, 4.1).

CHAPTER 2: ELEMENTS OF POTENTIAL FLOW

Simplifying approximation – the flow is ideal, i.e. no viscosity, incompressible, no surface tension effects, if it is a liquid, it does not vaporize.

2.1 Steady 2-D flow

We can specify a plane which has no velocity component perpendicular to it.

FIGURE

$q^2 = u^2 + v^2$

Continuity equation ideal for 2-D flow is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.1)$$

1.2 Stream function, ψ

FIGURE

Fix point A and allow point P to be variable. For steady incompressible flow for any point P, volume flow rate across AQP = that across ARP. No matter the shape of AQP, volume flow rate across it is constant.

Since A is fixed, the rate of flow across ARP is a function only of positive P, and this function is known as **Stream function, ψ**. The ψ_P is the volume flow rate across any line joining P to A. The value of ψ is arbitrarily set to zero at A.

FIGURE

If PP' is a streamline, rate of flow across AP is equal to rate flow across AP' since there is no flow across PP' (a streamline).

$\therefore \quad \psi_P = \psi_{P'}$

Flow may be represented by a series of streamlines at equal increment of ψ.

Consider PP' a small distance $\delta n \perp$ to streamline such that $AP'' >$ AP

Volume flow rate across AP'' is less than volume flow rate across AP by $\delta\psi$ across PP''. If the average velocity perpendicular to PP'' is q

$$\delta\psi = q\delta n$$

$$\Rightarrow \quad q = \frac{\partial\psi}{\partial n} \qquad (2.2)$$

Equation (2.2) shows that the closer the streamline for equal increament of ψ, the higher the velocty. $\psi = 0$ may be assigned to any convenient streamline.

Sign convection

FIGURE

The sign convection is that ψ increases from right to left when looking downstream. From equation (2.2) and this sign convection:

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}$$

FIGURE

At point P differentials of y and x related by

$$\frac{dy}{dx} = \frac{v}{u}$$

$$\therefore \quad vdx = udy$$

$$\text{or} \quad \frac{dx}{u} = \frac{dy}{v} \qquad (2.3)$$

Similarly for the x – z plane

$$\frac{dx}{dz} = \frac{u}{w} \quad \Rightarrow \quad \frac{dx}{u} = \frac{dz}{w} \qquad (2.4)$$

For the y-z plane

$$\frac{dy}{dz} = \frac{v}{w} \quad \Rightarrow \quad \frac{dz}{w} = \frac{dy}{v} \tag{2.5}$$

From equations (2.3) – (2.5)

$$\Rightarrow \quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{2.6}$$

The above equation is the mathematical definition of a streamloine. For a 2-D motion, we can relate the concepts of streamline to continuity equation. Continuty equation for 2-D flow is as stated in equation (2.1). This equation is satisfied automatically by introducing a new function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{2.7}$$

$$\psi = \psi(x, y, t)$$

Substituting equation (2.7) in (2.1)

$$\Rightarrow \quad \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Total differential of ψ:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Using equation (2.7)

$$d\psi = -vdx + udy \tag{2.8}$$

If $\psi = 0$ in equation (2.8), we obtain equation (2.3), which is the equation for a streamline.

Line of a constant stream function is called a streamline.

From equation (2.2)

$$d\psi = q\,dn$$

$\psi_2 - \psi_1$ = volume flow rate per unit width passing between the streamlines (m^2/s)

Note: We can define stream function in 2-D flow only, since there will be some ambiguity defining the third component of velocity in terms of stream function ψ in equation (2.7).

2.3 Circulation and Vorticity

Consider the figure below

FIGURE

Across any line AP in the fluid, the volume flow rate = $\int_A^P q_n \, ds$. Similarly, we can define $\int_A^P q_s \, ds$ along AP wholly in the fluid.

Integrating round a fixed closed circuit, we have

$$\Gamma = \oint q_s \, ds$$

This is called circulation which is positive for a counter clockwise direction.

Consider a rectangular element in the flow

FIGURE

$$\Gamma = u\delta x + \left(v + \left(\frac{\partial v}{\partial x}\right)\delta x\right)\delta y - \left(u + \left(\frac{\partial u}{\partial y}\right)\delta y\right)\delta x - v\delta y$$

$$\Gamma = \frac{\partial v}{\partial x}\delta x\delta y - \frac{\partial u}{\partial y}\delta x\delta y$$

$$Vorticity\ at\ a\ point\ = \frac{\Gamma\ around\ an\ infinitesimal\ circuit}{area\ of\ the\ circuit}$$

$$Vorticity,\ \omega\ = \frac{Circulation}{area}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (2.9)$$

Consider a small circular circuit of radius r

FIGURE

$$q_s = \Omega r$$

where Ω is the angular velocity

$$\Gamma = \oint q_s\, ds = \oint \Omega r \cdot r d\theta$$

$$= r^2 \oint \Omega d\theta$$

$$\Gamma = r^2 \cdot \overline{\Omega} \cdot 2\pi \qquad (2.10)$$

where $\overline{\Omega}$ = mean angular velocity for all particles on the circuitry about the centre

$$Vorticity,\ \omega = \frac{\Gamma}{A} = \frac{2\pi r^2 \overline{\Omega}}{\pi r^2}$$

$$Vorticity,\ \omega = 2\overline{\Omega} \qquad (2.11)$$

So, vorticity is twice the angular velocity. If vorticity is zero at all points in region then the flow in the region is said to be **irrotational**. Flow in region where vorticity is non-zero is said to be **rotational**.

Vectorial Approach

Consider the velocity vector at an elemental area of a control surface.

$\vec{n}\, x\, \vec{u} =$ vector tangential to ds of magnitude vector tangential to *ds* at magnitude $|\vec{u}|\, sin\ \theta$.

Vorticity at a point, ω

$$\omega = \lim_{\forall \to 0} \frac{1}{\forall} \int \vec{n}\, x\, \vec{u}\, ds$$

$$= \nabla \times \vec{u} = curl\, \vec{u} \cdots = \ldots local\ rotation \qquad (2.12)$$

$$curl\, \vec{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad (2.13)$$

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

Basic components of the motion of a fluid element

FIGURE

FIGURE

The deformation of the fluid element may be sub divided into two parts. The first consist of an angular motion of both sides through angle $1/2(\alpha - \beta)$. The second consists of an angular distribution $1/2(\alpha + \beta)$.

i.e. $1/2(\alpha - \beta)$ - rotation

$1/2(\alpha + \beta)$ - distortion

Rotation of the element through $1/2(\alpha - \beta)$ followed by distortion through $1/2(\alpha + \beta)$ leaves sides δx and δy in the angular position shown above.

Assuming α, β to be small.

$$\alpha = \frac{\partial v}{\partial x}\delta x\,\delta t\,\frac{1}{\delta x} = \frac{arc}{radius}$$

$$\beta = \frac{\partial u}{\partial y}\delta y\,\delta t\,\frac{1}{\delta y}$$

The average rate of rotation in the positive or counter clockwise sense

$$\Omega_z = \frac{1}{2}(\alpha - \beta)/\delta t$$

Fluid rotation is defined as the average angular velocity of two mutually perpendicular differential element of fluid.

Substituting for α, β

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (2.14a)$$

This is rotation about z-axis.

Similarly

$$\Omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad (2.14b)$$

$$\Omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (2.14c)$$

$$2\,\vec{\Omega} = \vec{\omega} = \nabla \times \vec{u} = vorticity$$

Note:

1. The rotation we are talking about is the rotation of an infinitesimal element about its axes and not the axis about which general rotational motion occurs in vortices or curvilinear flow.
2. Flows outside the boundary layer have almost no vorticity but those in the boundary layer have very strong vorticity.
3. For body to rotate there must be a torque applied by shear forces. Since there are no shear forces in inviscid flow, such flow is irrotational

1.4. **Potential Flow**

Irrotational flow is otherwise called potential flow.

$$\vec{\omega} = curl\,\vec{u} = \nabla \times \vec{u} = 0$$

A function whose curl is zero cam always be represented by the gradient of the scalar function Because of the vector identity

$$curl\left(grad\,\phi\right) = 0\,,$$

where ϕ = velocity potential

$$\vec{v} = grad\,\phi = \nabla\,\phi \qquad (2.15)$$

Continuity equation for incompressible flow

$$\nabla \cdot \vec{v} = 0 \qquad (2.16)$$

In view of equation (2.15)

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0 \qquad (2.17)$$

Equation (2.17) is called Laplace's equation

In Cartesian coordinate

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (2.18a)$$

In cylindrical polar coordinate

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (2.18b)$$

Any function ϕ which satisfies equation (2.18) can be a velocity potential representing an irrotational flow fluid

2-D Incompressible Potential Flow

In a 2-D flow potential function satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (2.19)$$

where

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \qquad (2.20)$$

For irrotational flow

$$\vec{\omega} = \nabla \times \vec{u} = 0$$

$$\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (2.21)$$

But $u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$ (2.22)

Substituting equation (2.22) in (2.21)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad (2.23)$$

Comparison of equation (2.20) and (2.22)

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial y} \qquad (2.24)$$

Equations (2.24) are known as Cauchy-Riemann's equations. From them, if either the potential or stream function is known the other may be computed. In 2-D polar coordinates the relation corresponding to equ. (2.24) is:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{\partial \psi}{r \partial \theta}, \quad v_\theta = \frac{\partial \phi}{r \partial \theta} = -\frac{\partial \psi}{\partial r} \qquad (2.25)$$

ϕ = constant lines are defined by

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

$$\left.\frac{dy}{dx}\right|_{\phi = const} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v}$$

ψ = constant lines are defined by

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$\left.\frac{dy}{dx}\right|_{\psi = const} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = \frac{v}{u}$$

Thus

$$\left(\left.\frac{dy}{dx}\right|_{\phi = const} \right) \left(\left.\frac{dy}{dx}\right|_{\psi = const} \right) = -1 \qquad (2.26)$$

which is the requirement that lines of constant ϕ and ψ are orthogonal.

FIGURE

Lines of constant ϕ and ψ form an orthogonal network.

From the spacing of ϕ and ψ lines, velocities can be computed. Pressure may be determined from Bernoulli's equation. Since there is no flow through any of the streamlines, anyone of them may be considered as possible solid boundary.

2.6 Flow Nets

Flow nets are grids of curvilinear square. These grids are formed by ψ = constant and ϕ = constant lines.

Procedure for drawing flow nets

- Fixed solid boundaries are streamlines since they have no flow across them.
- The axis of symmetry is also a streamline since there is no flow across it.
- Between the solid boundaries or axis of symmetry and solid boundaries other streamlines are sketch by guess work.
- The set ϕ = constant lines are drawn in such a way that they cross the streamlines at right angle making $\delta\psi = \delta\phi$.

The alternative to draw flow nets by trial and error is to use the precise mathematical expression for the stream function and velocity potential and plot the flow net exactly.

Flow Separation

Whenever divergence of streamline is appreciable the flow tends to separate, e.g. at the approach to a stagnation point A' in the figure below following recession of boundary B', and following sudden enlargement.

FIGURE

The more rapidly streamlines converge the better does the flow net (drawn on the basis that the flow is everywhere irrotational) represent the actual flow. In the zone of flow separation the solid boundary is not the outermost streamline. Thus flow net can indicate region in which separation may be expected in flow pass a given geometry and also how boundary may be 'streamlined' to reduce the chances of separation. Elimination of separation improves the flow pattern and reduces the dissipation of energy.

<u>Obtaining pressures and velocity from flow Nets</u>

Once velocity and pressure are specified at any point in a flow, velocity and pressure at any other point in the flow net can be obtained.

FIGURE

Flow rate between AB and CD remains constant

$$q_1 \Delta n_1 = q_2 \Delta n_2 \quad (2.27)$$

$$q_2 = q_1 \frac{\Delta n_1}{\Delta n_2} = q_1 \frac{\Delta s_1}{\Delta s_2} \quad (2.28)$$

$\Delta s_2 = \Delta n_2$ for square grids

Obtain pressure at any point by applying Bernoulli's equation.

$P + \frac{1}{2} \rho q^2 = P_o$ = constant throughout the flow for irrotational flow.

<u>Example 1:</u>

Check if the function $\psi = x^2 y^2$ represents a flow field. Sketch the field if it does.

<u>Solution:</u>

If ψ represents a flow field, the velocity components, u and v, derived from it must satisfy the continuity equation.

$$u = \frac{\partial \psi}{\partial y} = 2x^2 y$$

$$v = -\frac{\partial \psi}{\partial x} = -2xy^2$$

$$\frac{\partial u}{\partial x} = 4xy$$

$$\frac{\partial v}{\partial y} = -4xy$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4xy - 4xy = 0$$

Thus the given function ψ represents a flow field. It follows that

$$y = \pm \frac{\sqrt{\psi}}{x}$$

FIGURE

<u>Example 2:</u>

The components of the velocity vector of a flow field are $u = y$ and $v = -x$. Obtain the equation of the streamline.

<u>Solution:</u>

The equation of the streamline in two dimensions is

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$$-x\,dx = y\,dy$$

$$-\frac{x^2}{2} = \frac{y^2}{2} + c$$

$$x^2 + y^2 = c = r^2$$

This is an equation of a circle of radius r and centre at the origin.

FIGURE

Example 3:

Deduce the expression for velocity potential for the flow represented by $\psi = x\,y$

Solution:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = x$$

$$v = -\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial y} = -y$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$= x\,dx - y\,dy$$

$$\phi = \int x\,dx - \int y\,dy$$

$$\phi = \frac{x^2}{2} - \frac{y^2}{2} + c$$

$$\phi = \frac{x^2 - y^2}{2} + c$$

where c is a constant of integration. In order to eliminate c, ϕ can be zero value at the origin. Thus $\phi = \dfrac{x^2 - y^2}{2}$.

Example 4:

Using the orthogonality relationship between ϕ and ψ, deduce the general equation for the gradient of the velocity potential for an irrotational flow represented by $\psi = x - x^2 + y^2$.

Solution:

$$\frac{\partial \psi}{\partial x} = 1 - 2x$$

$$\frac{\partial \psi}{\partial y} = 2y$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$(1 - 2x)dx + 2y\,dy = 0$$

$$\left.\frac{dy}{dx}\right|_{\psi} = -\frac{(1 - 2x)}{2y}$$

Using the relation $\left(\left.\dfrac{dy}{dx}\right|_{\phi}\right)\left(\left.\dfrac{dy}{dx}\right|_{\psi}\right) = -1$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\phi} = -\frac{2y}{(1 - 2x)}$$

Vortices

Energy variation across curved stream lines

FIGURE

Two streamlines on an inclined plane

α = angular of plane to horizontal

Consider steady flow between two curved streamlines. P and V will be functions of *r* and θ but if flow is assumed to be axisymmetric ($\partial/\partial\theta = 0$) then P and V will be function of *r* only.

Pressure force acting radically outward

Pr, Sø – (P + $^{2p}/_{2r}$Sr)(r + Sr)Sø + 2(P + $^{1}/_{2}$$^{2p}/_{2r}$Sr)Sr Sø

Simplifying and ignoring third order term we obtain – $^{2p}/_{\partial r}$ r St Sø

Weight forces

Component of weight in outward radial direction is – pgr ∂ø Sr sin α = pgr ∂2Sø

Since ∂r sin α = ∂2

The total of the two forces create the required centripetal acceleration.

→ - ($^{\partial r}/_{rr}$Sr∂ør - pgr∂z∂ø) = pr∂ø∂r $v^2/_r$

Simplifying, we obtain

dp/dr + pg dz/dr – pv2/v = 0 …………………………………………………..(29)

therefore dp = -pgdz + pv2/rdr

or dp/pg = - dz + v2/grdr ……………………………………………………………(30)

Bernoulli's equation states

E = z + p/pg + V2/2g ……………………………………………………………………(31)

This is constant along a streamline if flow is frictionless. Differentiating equation (31)

dE/dr = dz/dr + 1/pg dp/dr + v/g dv/dr ……………………………………………..(32)

From equation (30)

dp/dr = - pg dz/dr + p v2/r ……………………………………………………….….(33)

substituting equation (33) in (32))

dE/dr = dz/dr + 1/pg (-pg dz/dr + p v2/r) + v/g dv/dr

dE/dr = v/g(v/r + dv/dr) …………………………………………………………………………(34)

This is the expression for the variation of total energy along streamlines included to the horizontal. It also applies to streamlines on the horizontal.

<u>Two-Dimensional Curvilinear flow of inviscid flow</u>

Two-dimensional flow may involve

Free or Natural cylindrical vortex {both have zero verticity }

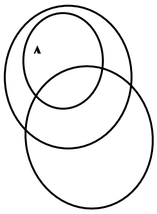
Free or Natural Spiral vortex {both have zero verticity }

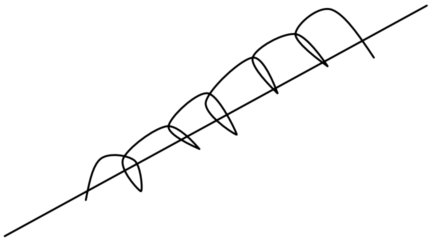
Force4d vortex

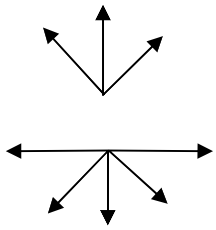
Radial flow

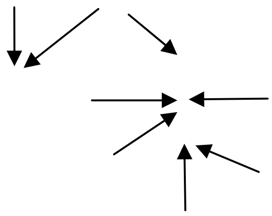
A vortex is a mass of fluid in which the flow is circulating.

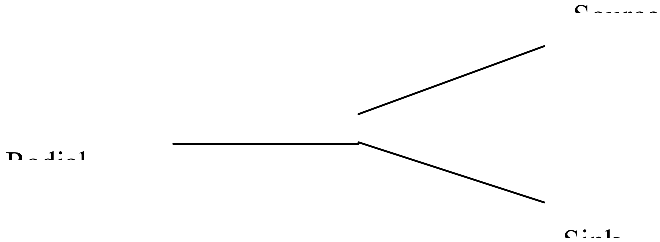
Filament of vortex is the locus of the centres of circulation











A sources is a point within a fluid from which fluid issues out symmetrically in all directions. Strength q of a sources is the volume of fluid which issued from it in a unit time →$V_r = q/2\pi r$.

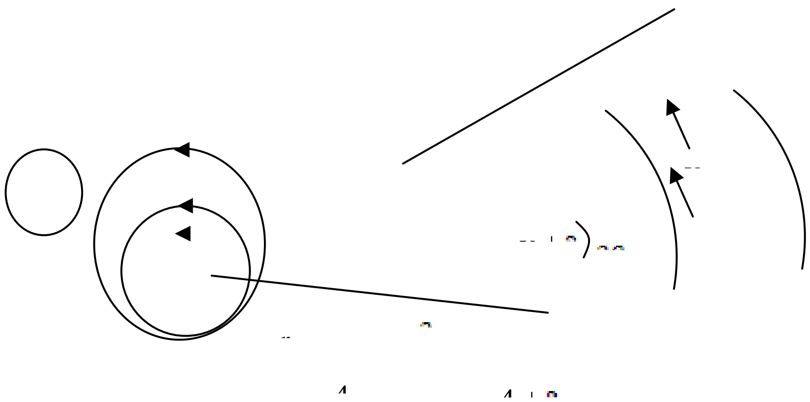
A sink is the exact opposite of a sources, i.e a point to which the fluid converges uniformly and from which fluid id continuously removed.

Note:

A sowrie or sink is an abstraction which can never be perfectly realized. But it is a useful mathematical concepts.

<u>Free cylindrical vortex (a.k.a. potential vortex)</u>

A flow pattern in which the streamlines are concentric circles is known as a plane circular vortex if the particles of fluid in the flow do not rotate on their own axes, the flows is said to be irrotational and the vortex is known as an irrotational or free vortex.



The circulation around the element area $r\partial ø. \partial r$ is Γ (+ve ccw) = $(v + \partial v)(r + \partial r)\, \partial ø - vr\, \partial ø = (r\, \partial v + v\partial v)\, \partial ø$.

Vorticity, $w = \Gamma/\text{area} = v/r + dv/dr$ as $\partial r \rightarrow 0$

For irrotational flow, w is zero

i.e. $v/r + dv/dr = 0$

integrating the expression

$\rightarrow \text{Ln } V + \text{Ln } r = A$

$\text{Ln}(vr) = A$

$e^{\text{LnVr}} = e^{A}$

$Vr = c$

$V = c/r$ ……………………………………..(34)

C is the strength of the vortex. This equation describes the variation of velocity with radius in a free, irrotational vortex.

circulation around a circuit corresponding to a streamline

$\Gamma = 2\pi rv$

But vr = constant

$\rightarrow \Gamma$ = constant

$\pi W = \Gamma/area = 2\pi rv/\pi r2 = 2v/r = 0.$

The centre of free vortex is rotational, the velocity there tends to infinity as well as velocity centre is a singular point.

Horizontal variation of pressure

Applying Bernoulli's equation to any two concentric streamlines on a horizontal plane.

$P_1/Pg + V_1^2/2g = P2/Pg + V^2/2g$

$P_1 - P_2/Pg = V_2^2 - V_1^2/2g$

Applying equation ………… (34)

$V_1 = C/r_1$, $V_2 = C/r_2$

$\rightarrow P_1 - P_2/Pg = C^2/2(1/r_2^2 - 1/r_1^2)$ …………………(35)

For compressible flow

$P_1 / P_1 - P_2 / P_2 = C^2/2\ (1/r^2/_2 - 1/r^2{}_1$ ……………… (36)

Variation of pressure, P with height and rading from equ. (30), $dp = - pgdz + PV^2/r\ dr$

Applying equ. (34)

$Dp = - pgdz + pc^2/r^3\ dr$

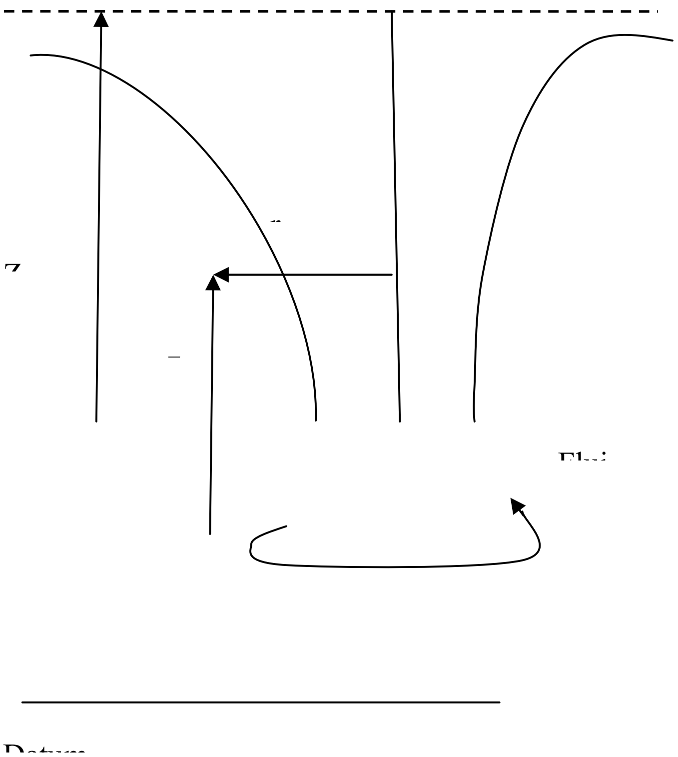
Integrating: $P = - pgz - pc^2/2r^2 + B$, Boundary conditions:

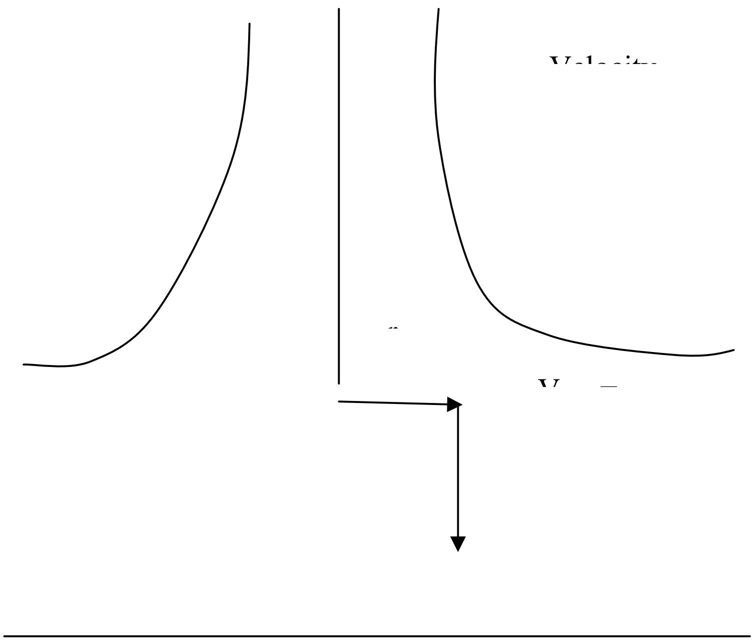
$P=P_o$ as $r \rightarrow w$ and $z = z_o$ then $B_1 = P_o + pgz_o$ $\frac{P - P_o}{P} = g\ (z_o - z) - C_2/2r^2$ …………..(37)

For flow with a free surface, P=Po on the free surface and equ. (37) becomes $z_s - z_o - C^2/2gr^2$ …………..(38)

Where zs = height of free surface at radins r. equ. (38) is the equation of a second order.

HYPERBOLOID





(ii) **Forced vortex**

This is a vortex in which the fluid instates as a solid body about an axis due to extreme force.

Let it be the angular velocity.

V = wr ……………….. (39)

W = v/r = dv/dr ………………… (39a)

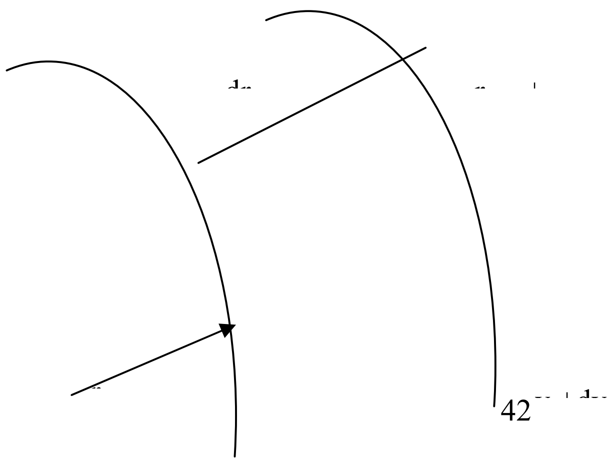
From equ. (34)

De/dr = v/g (dv/dr + v/r) = v/g (2 w)

P^2,de= ∫ v/g (2w)dr = $\int^2_1$ 2 w^2r/g dr

E_2 - E^1 = w^2/g ($r_2^2 - r_1^2$

$$E2 - E1 = \frac{V^2_2 - V^2_1}{g} \quad \text{……………… (40)}$$



Verticity = $\frac{v}{r} + \frac{dv}{dr} = 2w$

From equ.(30)

$dp = -\,pgdz + pv^2 \frac{dr}{r}$

Applying equ. (39)

$dp = -pgdz + Pw^{-2}rdr$

Integrating

$P = -\ pgz + p\frac{w^{-2}r^2}{2} + B$

Boundary conditions:

$P = P_o + r = r_o$ and $z = z_o$

$= B = Po + pgz_o - p\frac{w^{-2}r^2_o}{2}$

$P - Po = 1/2w^{-2}(r^2 - r^2_o) - g(z - z_o)$ …………… (41)

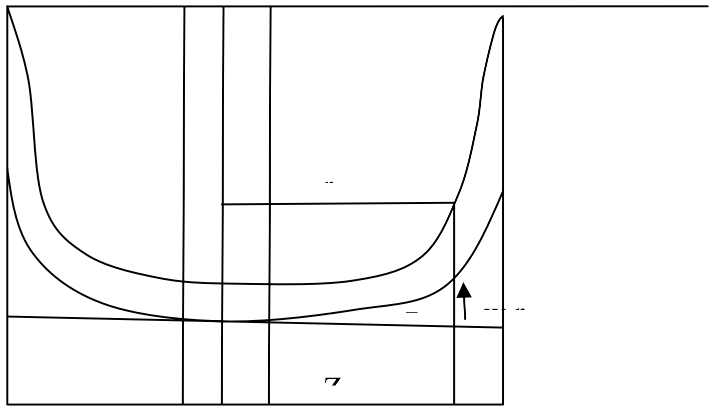
If vortex has a free surface P = Po = constant at the free surface

From equ (41) the profile of the free surface is given as

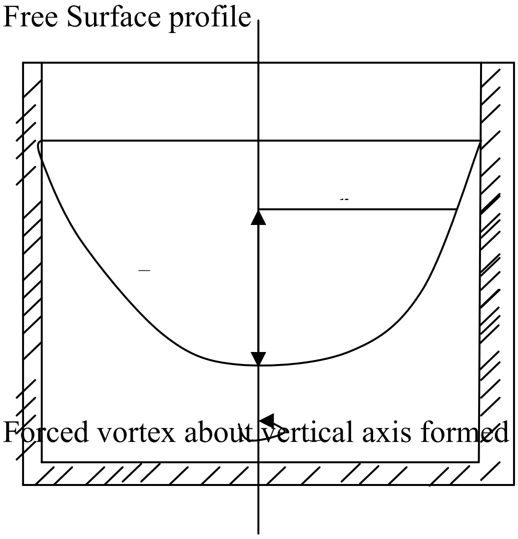
$Z_s = z_o + w^{-2}(r^2 - r_0^2)$

2g

Which is the equation of a PARABOLOID



Free Surface profile



Forced vortex about vertical axis formed in liquid in open container

(iii) **Combined vortex (Ramkine vortex)**

Forced vortex at core surrounded with free cyclical vortex. Let r=a be the Unit of forced vortex core.

The velocity at r=a is $v = w^{-} a$

Velocity of the free vortex at r = a is v = c/a

These two velocities values are the same: $w^{-} = c/a^2$ ……………. (42)

At $r = a$, using (41) pressure is

$P_a = {}^1/_2\, pw^{-2}a^2 - pg\,(z - z_0) + P_o$ ………………. (43)

Using the equation derived for free vortex

$P_1 = -\, pgz - {}^{pc2}/_{2r2} + B_1$ ……………(44)

At a,

$B = P_a + pgz + pc^2/2a_2$

But $V = c/a = w^{-}a$

$\rightarrow c^2/a_2 = w^{-2}a^2$

$B_1 = P_a + pgz + p/_2\, w^{-2}a^2$ …………………(45)

Substituting for P_a from equ. (43).

$\rightarrow B1 = ½\, pw^{-2}a^2 - pgz + pgz_0 + P_0 + ½\, pw^{-2}a^2$

$= pw^{-2}a^2 + pgz_0 + p_0$ ……………(46)

Substituting equ. (46) in equ. (44) the pressure distribution at $r > a$ is

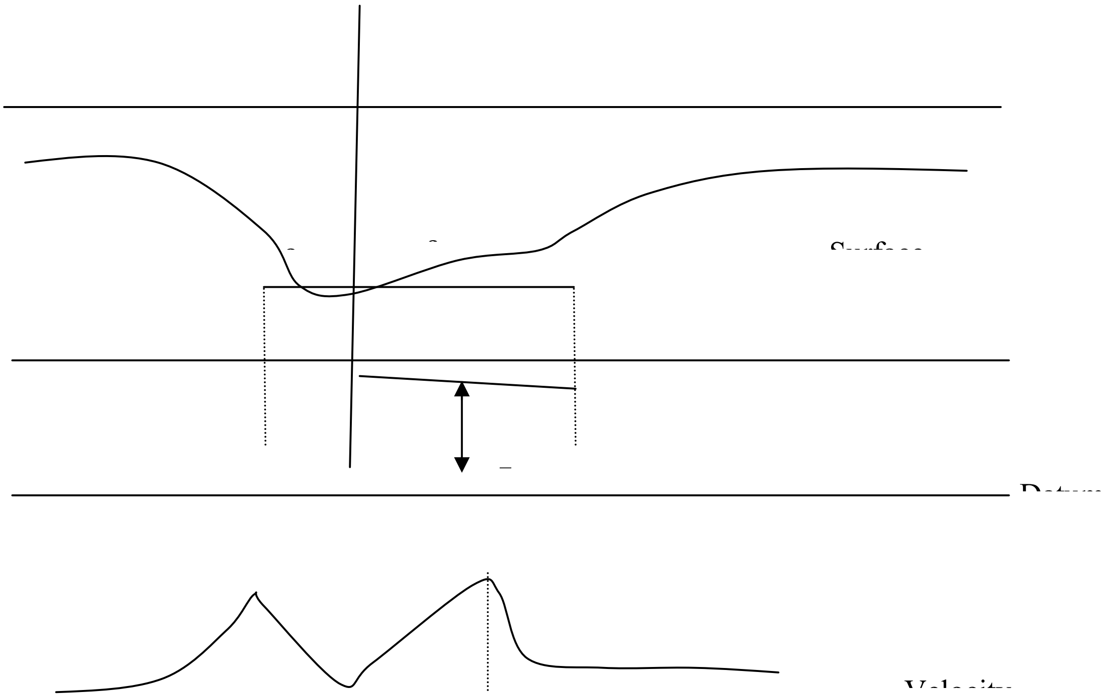
$P = -pgz - P\, c^2/2r^2 + pw^{-2}a^2 + pgz_0 + p_0$

Substituting for a from equ. (42)

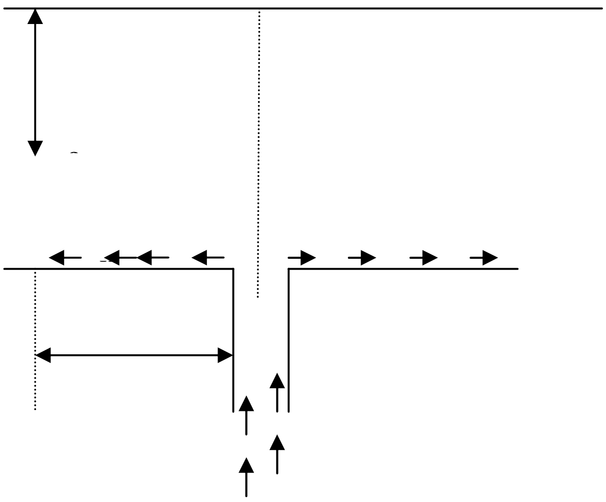
$P = P_0 + pg(z_0 - z) - p/2\, w^{-2}a^4/r^2 + pw^{-2}a^2$

$P = P_0 + pw^{-2}a^2\,(1 - a^2/2r_2) - pg\,(z - z_0)$. …………………… (47)

If there is a free surface $P = p_0$ at the free surface, the free surface equ. For $r>a$ is $z_s = z_0 + w^{-2}a^2/g\,(1 - a^2/2r^2)$



(iv) **Radial flow**



Let the velocity at distance x from the centre be vx. The horizontal flow can be considered as curvilinear flow of infinite radins.

Equation derived for free vortex may then be used.

$$Vr = c$$

Differentiating

$Vdr + rdv = o$

$v/r = - dv/dr$

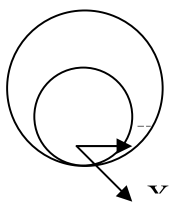
$de/dr = v/g\ (- dv/dr + dv/dr) = o$

No change in energy across the streamline.

Q = source strength = m^3/s

Q = volume flow rate

$= v = q/2rxt$



$v_1 = q/2\prod x_1 t$　　　　$v_2 = q/2\prod x_2 t$

Applying Bcrnoulli's equation between any two points

$P_2/pg + v^2{}_2/2g + z_2 = P_1/pg + v^2{}_1/2g + z_1$

$P_2 - p_1/pg = q^2/8\prod^2 t^2 g\ (1/x21 - 1/x22) + (z_1 - z_2)$ ……………….(48)

(v) Free spiral vortex

This is a combination of free cylindrical vortex and radial flow.

Applying Bcrnoulli equation between any two points (since E is constant).

$p_1/pg + v^2{}_1/2g + z_1 = P_2/pg + v^2{}_2/2g + z_2$

But $v^2{}_2 = v^2{}_{2t} + v^2{}_{2r}$ and $v^2{}_{1t} + v^2{}_{1r}$

So $\frac{P_2 - P_1}{pg} + (z_2 - z_1) = \frac{v^2{}_{1t} - v^2{}_{2t}}{2g}$ r $\frac{v^2{}_{1r} - v^2{}_{2r}}{2g}$

Considering unit thickness of the flow.

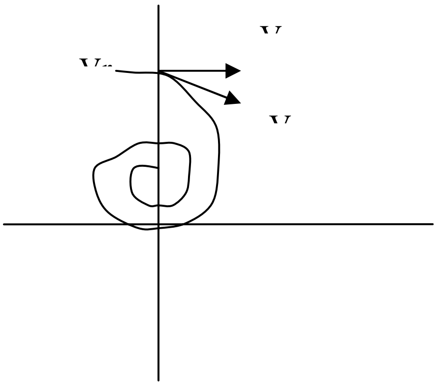
$V_{1r} = Q/2\prod r_1$ and $v_{2r} = Q/2\prod r_2$

Where Q = flow rate per unit thickness

$V_{1t} = c/r_1$ and $v_{2t} = c/r_2$

$\frac{P_2 - p_1}{Pg} + (z_2 - z_1) = \frac{c^2 + (Q/2\prod)^2}{2g} (1/r^2{}_1 - 1/r^2{}_2)$ ……………..(49)

$V_{1t}/v_{1r} = v_{2t}/v_{2r} = 2\prod c/Q$ ………………… (50)



$\theta = \tan^{-1} v_t/v_r$

= constant for all radii

Path of a fluid particle passing through such a vortex is an EQUIANGULAR SPIRAL.

Examples of free spiral vortex are bath tab vortex, the tornado, etc.

<u>Flow pattern and their combination potential and stream function for simple flow</u>

<u>Uniform flow parallel to the x-axis</u>

Q = Ax satisfies lapcore equation.

$U = \partial\varphi/\partial x = A$

$V = \partial Q/\partial y = 0$

Using Cauchy – Rieman's equation

A = ∂φ/∂y

φ = Ay + f(x)

Where f(x) is an arbitrary function of x

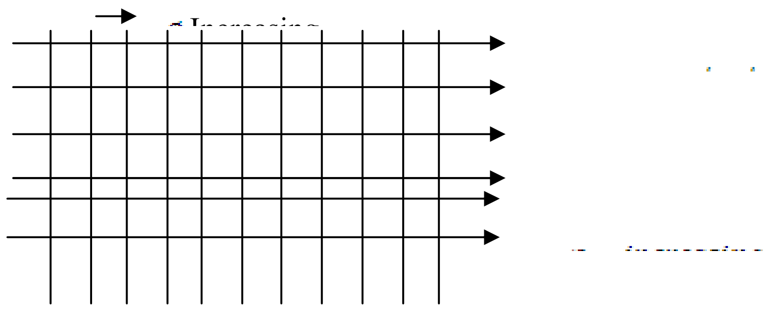
But ∂φ/∂y = 0 = - ∂φ/∂x = df/dx

:- f = constant

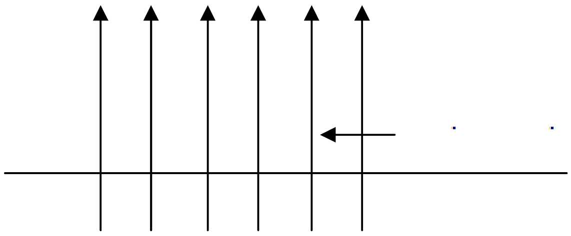
Take f = 0, for commence

→ φ = Ay …….. (51)

Q = Ax ………….(52)

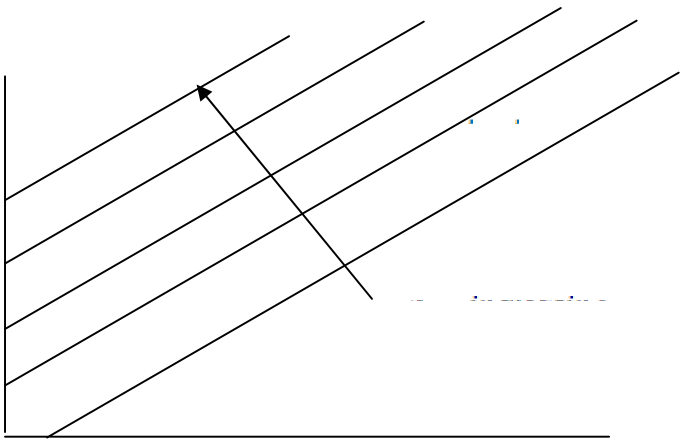


Uniform flow parallel to the y-axis flowing upward

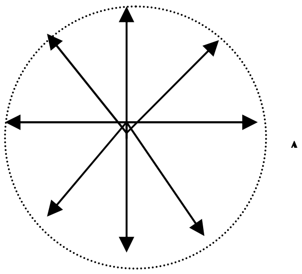


Uniform flow not parallel to any of the axes

$\varphi = Ay - Bx + C$



Line sink



$0 \leq \theta \leq 2\pi$

$Vr = q/2\prod r$

Where q = strength of source

$Vr = 0$

$Vr = \partial\varphi/\partial r = 1/r\ \partial\varphi/r\theta$

$V\theta = 1/r\ \partial\theta/\partial r = -\ \partial\varphi/\partial r$

$\varphi_p - \varphi_A$ = flow across Ap

$= \int^{\theta}_{0} vr.rd\theta = \int_{0}^{\theta} q/\partial\prod\ .\ rd\theta$

Let $\varphi_A = 0$, at $\theta = 0$, $\varphi = 0$

$\varphi = q/\partial\prod\ \theta$ ………………….. (55)

$vr = \partial\varphi/\partial r$

ɖ $- \varphi_o = \int^{r}_{o} vr\ dr$

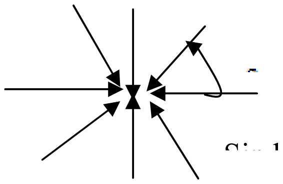
$= \int^{r}_{o} q/2\prod\ dr$

= q/2∏ en (r/r_o)

Let φ = 0 at $r_o = 1$

Then φ = q/2∏ enr …………… (56)

Flow to a line sink



θ = - q$\theta/2\pi + c$

ϕ = -q$/2\pi \ln r$

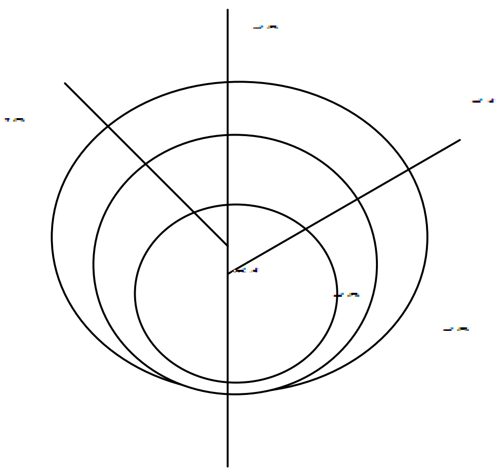
Vortices

Free vortex

Vr = O, vθ = c/r

Where c = strength of vortex

= ŋ/2∏



$d\varphi = v_r r d\theta - v\theta\ dr$

Since vr = 0

$d\varphi = - v\theta\ dr.$

$\varphi = - \int v\theta\ dr = - \int c/r\ dr$

$$= - \frac{\Gamma}{2\Pi} \int \frac{1}{r} dr$$

Let $\varphi = 0$ at r = 1

Then $\varphi = \frac{\Gamma}{2\Pi}$ enr ……………. (58)

For ccw rotation

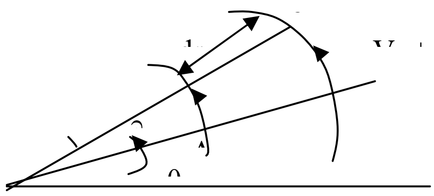
Note that the sign of these equation becomes positive for cw rotation, since Γ = - ve for cw rotation.

$d\varphi = vr\ dr + r v_\theta\ d_\theta$

upon substitution and making $\varphi = 0$ at $\theta = 0$

$\varphi = \int \Gamma/2\Pi\ d\theta = \Gamma\theta/2\Pi$ ………….. (59)

(ii) **FORCED VORTEX**



Γ ABCD = $(V_\theta + d\,V_\theta)(r + dr)d\theta - V_\theta r d_\theta$

$=(V_\theta dr + rdV_\theta)\, d_\theta$

Dividing by area

Vorticing of = 2ϖ

$V_{\theta} = \varpi r$, $\quad$ $Vr = 0$

$d\varphi = d\varphi/ d\theta + d\varphi/ dr\, dr$

$= V_r r\, d\theta - V_\theta\, dr$

$\varphi = - \int v_\theta dr = - \int w^{-}rdr = - \frac{1}{2} w^{-}r^2 + \text{const.}$

but $\varphi = 0$ at $r = 0$

:- $\varphi = - \frac{1}{2} w^{-}r^2$

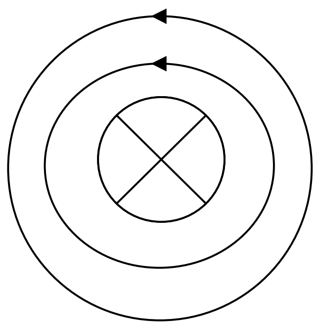
For counter clockwise rotation.

If vortex is cw

$\varphi = \frac{1}{2} wr^2$

(iii) **RAMKINE VORTEX**

Rotational core of rations a



$\Phi = -\Gamma/2\prod \ln (r/a)$

$\Phi = 0$ at $r = a$

Compiled by Prof. M. A. Waheed

Super position

The governing equation for potential flow $v^{-2}ø = 0$

Linear partial differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

Let φ_1 and φ_2 represent two different solution of lapcole's equation. Then if

$\varphi_3 = \varphi_1 + \varphi_2$, φ_3 is also a solution since

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} =$$

$\rightarrow 1$ $\qquad\qquad$ $\rightarrow 2$

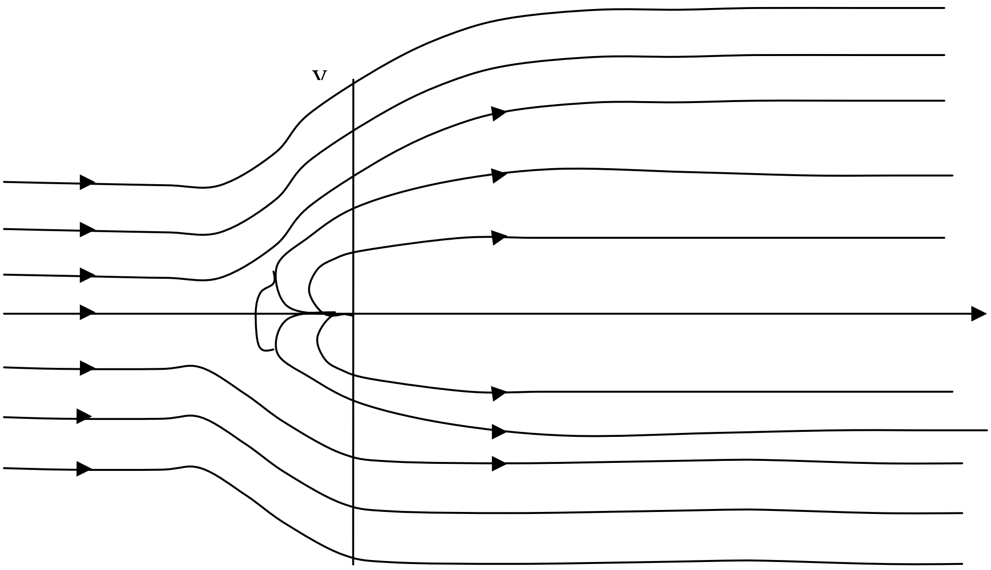
Likewise velocity given by φ_1 and φ_2 can also be added vertorially to get velocity given by φ_3. But pressure corresponding to φ_1 and φ_2 cannot be added to give pressure of φ_3 since Bernoulli equation is not linear in velocity terms.

Combination of Basic flow pattern

(a) Uniform rectilinear flow + line source.

$$\varphi = uy + \frac{q\theta}{2\Pi} = ur \sin\theta + \frac{q\theta}{2\Pi}$$

Uniform flow with velocity parallel to x-



S = stagnation point at s

Velocity = 0

Pt (r,θ) = (q/2∏v , ∏)

φ = ursinθ + q θ/2∏

vr = 1/r ∂φ/∂θ

vr = 1/r (ursinθ + q/2∏)

= ucosθ + q/2∏

At s, vr = 0

:- cosθ = - q

2∏ru

Vθ = - ∂φ/∂r

= usinθ

At s, vθ = 0

:- usinθ = 0

$\sin\theta = 0$

:- $\theta = \prod$

At s, $v\theta = vr = 0$

Combining both results at s, $\theta = \prod$

$$\cos\prod = -\frac{q}{2\prod ru}$$

$$-1 = \frac{-q}{2\prod ru}$$

$$\rightarrow r = \frac{q}{2\prod ru}$$

Hence coordinate s $(r,\theta) = \left(\frac{q}{2\prod ru}, \prod\right)$

Substitute in equ. (60)

$\varphi = ur\sin\theta + q\theta/2\prod$

at staymation point

$\varphi_s = u, q/2\prod u \, sm\prod + q\prod/2\prod$

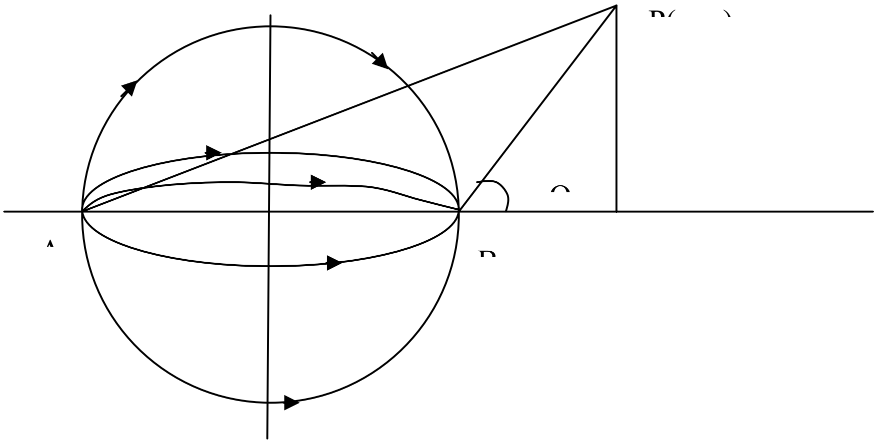
$\varphi_s = q/2$

The streamline $\varphi = q/2$ can represent a solid boundary since there can be no flow across a streamline. This streamline is called **RANLINE BODY** or **HALF-BODY**.

As r → w, u = U, v → = O

Source and sink of numerically equal strength

Source at A and sink at B



$\varphi = q\theta_1/2\prod - q\theta_2/2\prod = q/2\prod (\theta_1 - \theta_2)$ ……….(61)

If A is at (-b, θ) and B is at (b, O)

Then $\theta 1 = \tan^{-1} y/x+b$ $\theta 2 = \tan^{-1} y/x-b$

$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \frac{y/x+b - y/x-b}{1 + (y/x+b)(y^2x-b)}$$

$= -2by/x^2-b^2+y^2$

$$\Rightarrow \theta_1 - \theta_2 = \tan^{-1} = \left(\frac{-2yb}{x^2-b^2+y^2}\right)$$

$$\varphi = -q/2\prod \tan^{-1} = \left(\frac{-2yb}{x^2-b^2+y^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ (62)$$

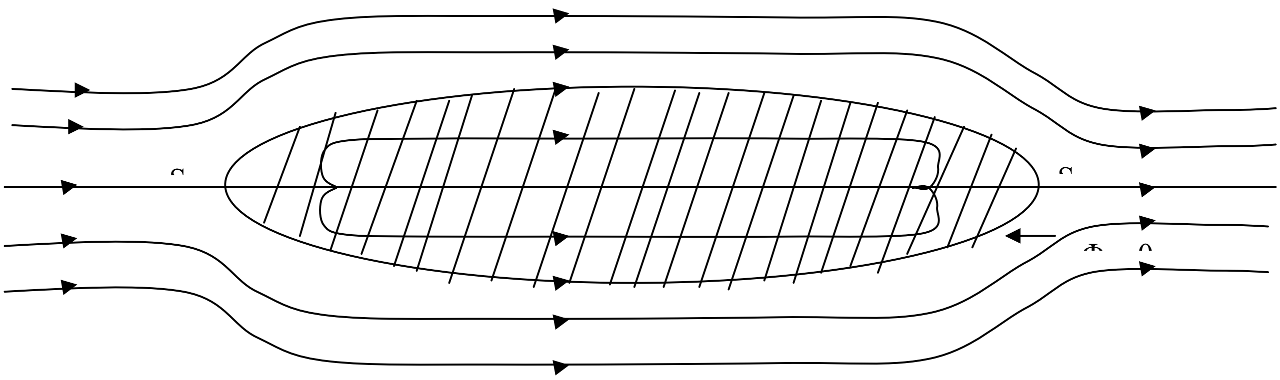
(c) **Source and sink of numerically equal strength combined with uniform rectilinear flow**.

Uniform rectilinear flow with velocity U parallel to x-axis.

Source at (-b,O) and sink at (b,o)

$$\varphi = Uy - q/2\prod (\theta_2 - \theta_1)$$

$$= Uy - q/2\prod \tan^{-1} = \left(\frac{-2yb}{x^2-b^2+y^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(63)$$

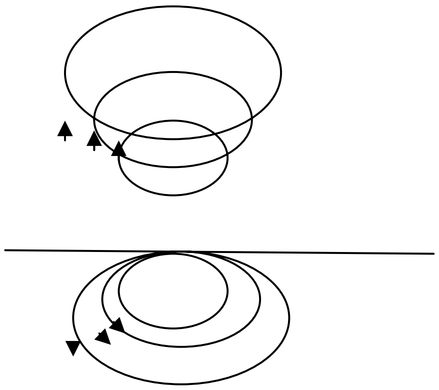


Stagnation pts s_1 and s_2

Line φ = O called Ranline oval can replace the oval with a solid body of that shape.

(d) <u>Doublet or Dipole (Source + sink)</u>

If the source and the sink shown in (b) are moved indefinitely closer together but the product qx2b is maintained firute and constant the resulting pattern is a doublet or dipole. Angle apis becomes zero and the streamline becomes circle tangent to the x-axis.



2bq = constant

As 2b → 0

$\tan \theta \to 0$

From equation ………… (62)

$$\Phi \rightarrow -\ 1/2\prod \left[\frac{(2bq)y}{x2-b2+y2}\right] \rightarrow -\ \frac{cy}{x2+y2} = -\frac{cr\sin\theta}{r2} +$$

$$\Phi = -\ \frac{cr\sin\theta}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(64)$$

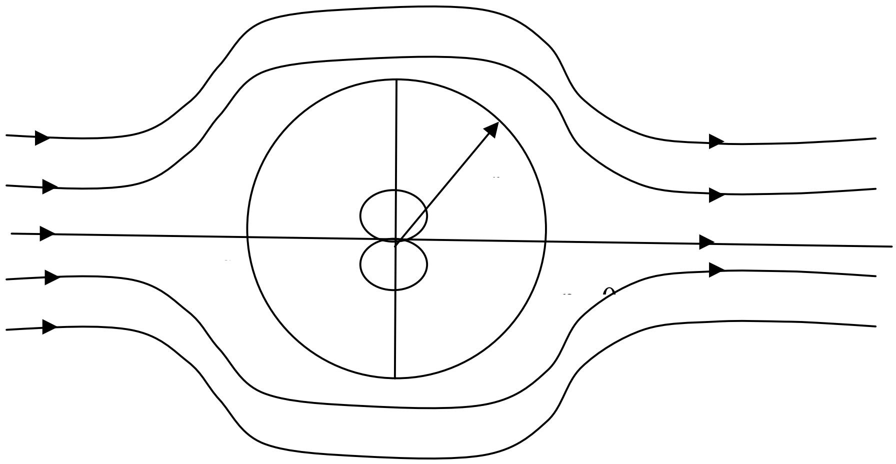
Where cr = qb/π = constant = strength of doublet

(e) Doublet + uniform rectilinear flow

$\Phi = u_y - \frac{cr\sin\theta}{r}$

$= ur \sin \theta - \frac{cr\sin\theta}{r}$

$\Phi = (ur - c/r) \sin \theta$ ……………………(65)



Equation … (65) show that $\Phi = 0$ when $\theta = 0$, π or $c = ur2, r = \frac{\sqrt{c}}{u}$

Let $c/u = a^2$

i.e $\Phi = 0$ along x – axis and a radius r = a about origin equation (65) may be written as

$\Phi = u(r - a^2/r)\sin\theta$ …………………….. (65a)

This flow pattern represents 2 – D ideal in-viscid flow around a circular cylinder of radius a with its centre at the origin.

Velocity at any point in the flow

$v_r = 1/r\ 2\Phi/2\theta = u(1 - a^2/r^2)\cos\theta$

$v_2 = -\ 2\Phi/2\theta = u(1 - a^2/r^2)\sin\theta$

At the surface of the cylinder

r = a, and $v_r = 0$, $v_2 = -2u\sin\theta$

Stagnation point labeled s occur at $\theta = 0$ and $\theta = \pi$.

Velocity at surface is a maximum at $\theta = \frac{\pi}{2} and\ \theta = \frac{3}{2\pi}$

$V = \sqrt{vr^2 + v\theta^2}$

Far upstream where P is unaffected by cylinder, piezometric pressure, = P = P_p + pgz

Applying Bernoullis equation.

$P = ½ pu^2 = p^x + ½ pv^2$

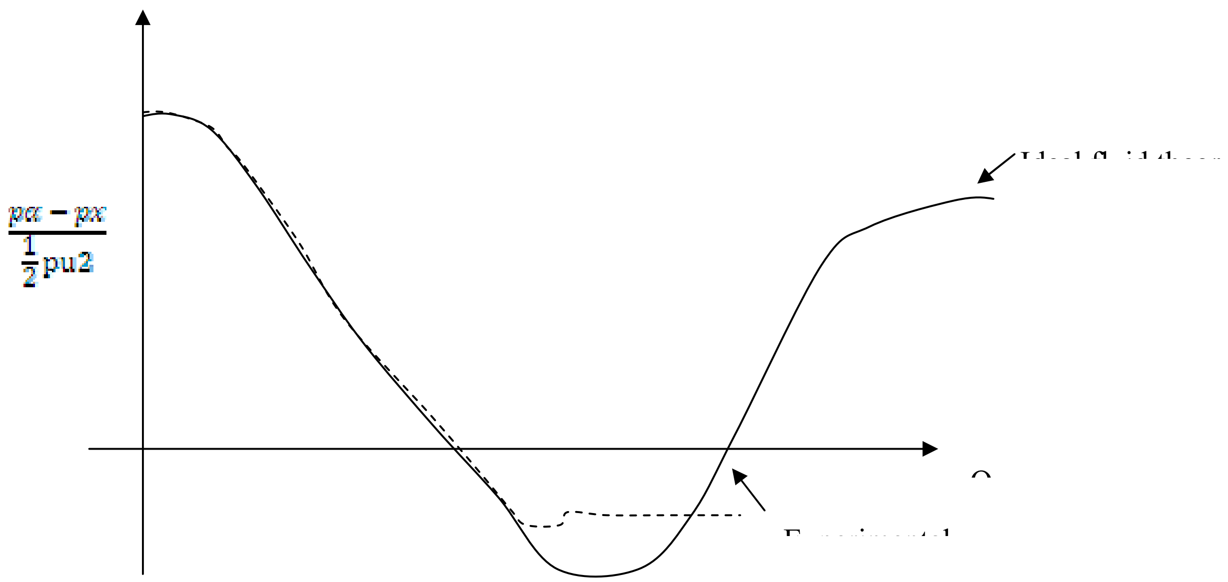
$P = P + ½ pu^2 - ½ pv^2$

At cylinder surface, $v = v_\theta = - 2u\sin\theta$

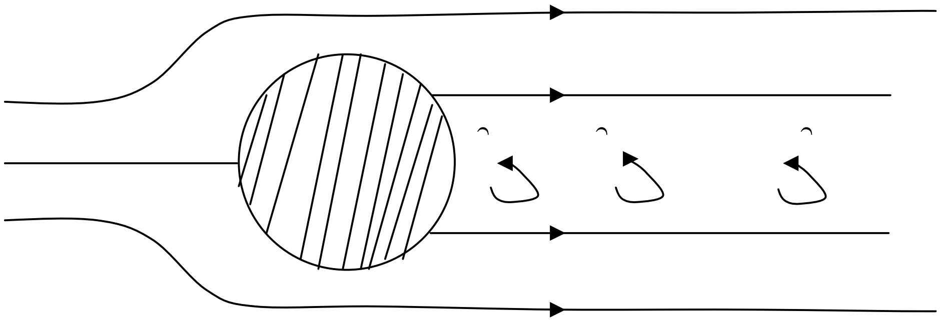
$P - P = ½ pu^2 - ½ p. 4 u^2 \sin^2\theta$

$\quad = ½ pu^2 (1 – 4 \sin^2\theta)$

$\frac{P – p}{½ pu^2} = (1 – 4 \sin\theta)$



The pressure distribution is symmetrical about the origin, hence the net force exerted by the fluid on the cylinder in any direction is zero capart from a possible huvy only force). This result conflicts with practical experience, and the condition is known as d'Alembert's paradox.



Real flow around a cylinder (Laminar flow)

(f) **Free spiral vortex**

This is the combination of free cyclical vortex and radial flow.

Source + cw vortex

$\Psi_{sv} = \psi$ source $+ \psi$ free vortex

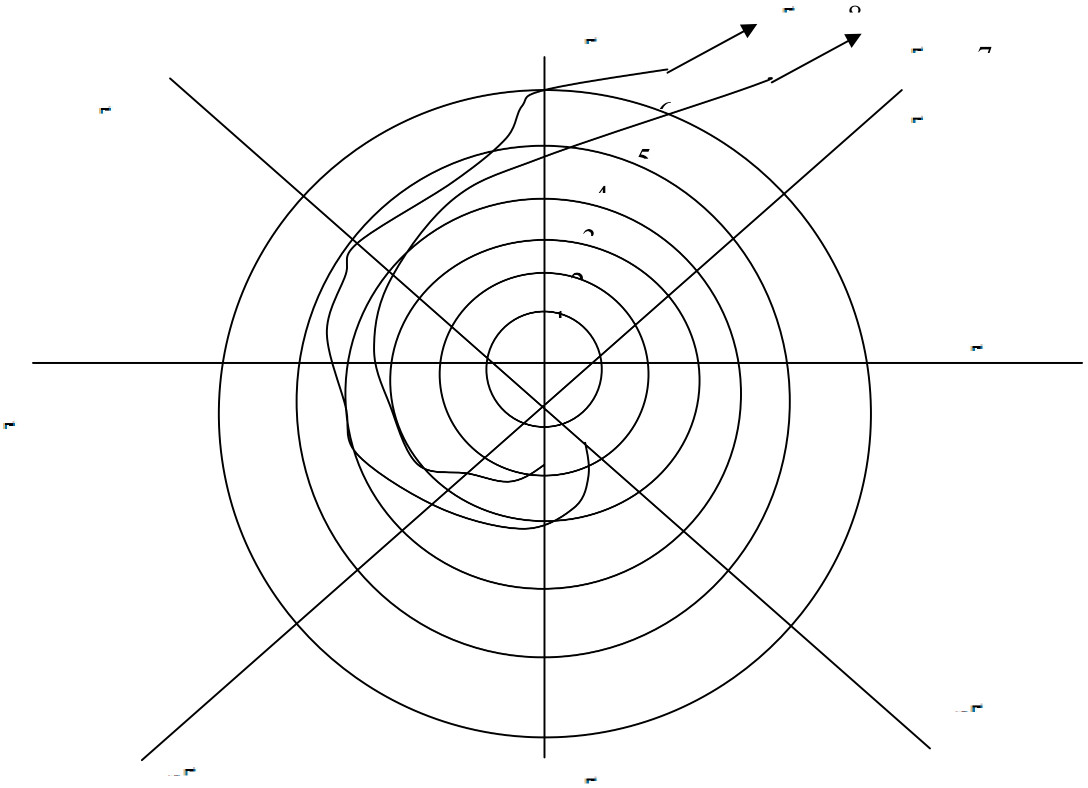
$= \frac{q\theta}{2\Pi} + \Gamma/2\Pi\ enr$

$= 1/2\Pi\ (q\theta + \Gamma\ enr)$ ……………………(66)

$\Psi_{sv} = \psi_{source} + \psi_{free\ vortex}$

$= \frac{q\theta}{2\Pi}$ enr + $\Gamma/2\Pi\ \theta$

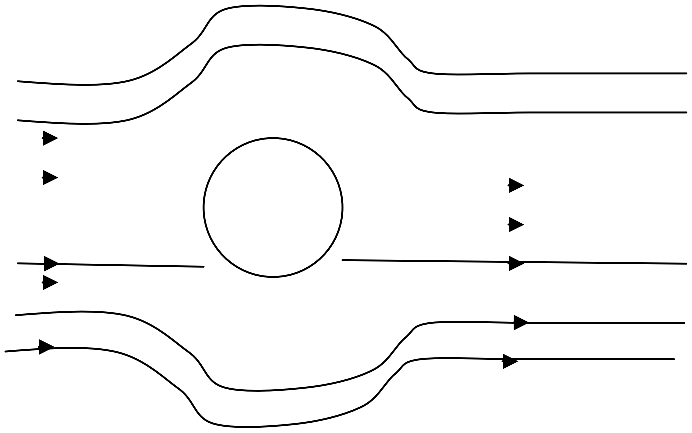
$= 1/2\Pi$ (q enr $+ \Gamma\ \theta$) ……………………(67)



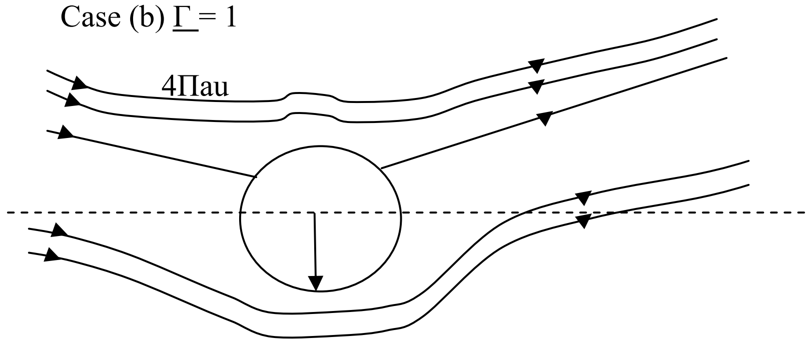
(g) **Doublet, uniform rectilinear flow and free vortex**

$\Psi = u\,(r - a2/r)\sin\theta - \Gamma/2\Pi$ en (r/r_o) …………….(68)

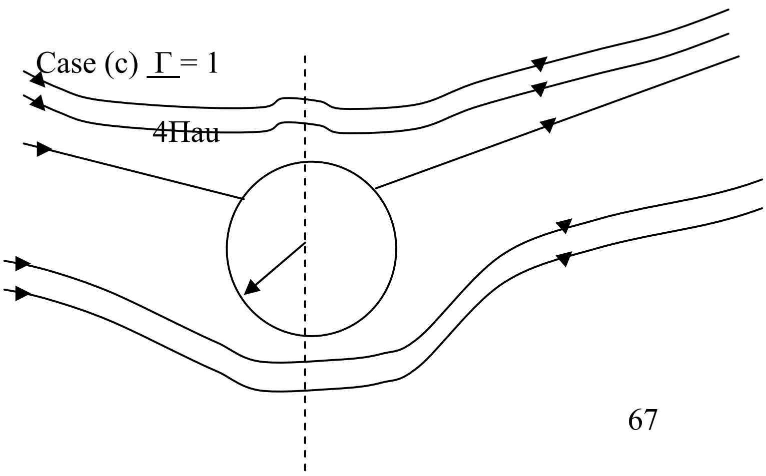
Case (a) $\Gamma = 0$



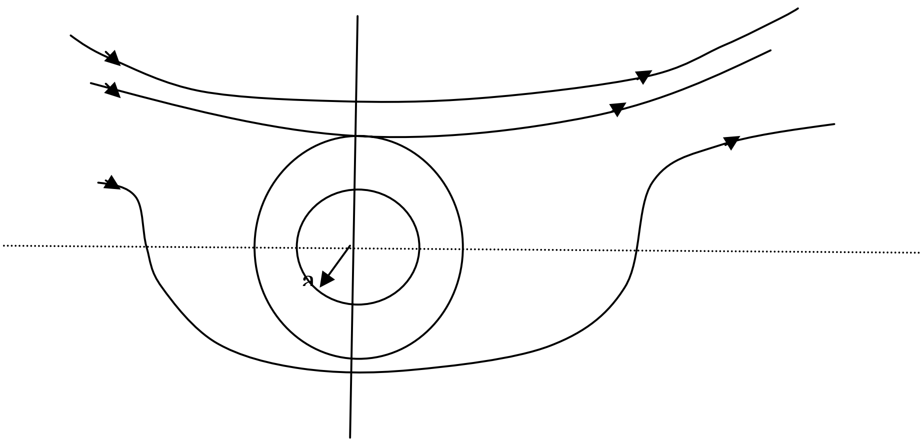
Case (b) $\frac{\Gamma}{4\Pi au} = 1$



Case (c) $\frac{\Gamma}{4\Pi au} = 1$



Case (d)) $\frac{\Gamma}{4\Pi au} => 1$



Tangential velocity

$$v_{\theta} = -\frac{\partial \Psi}{\partial r}$$

$$= -\{u(1 + a^2/r^2)\sin\theta - \Gamma/2\Pi \,.1/r\}$$

$$= -u(1 + a^2/r^2)\sin\theta + \Gamma/2\Pi \,.1/r$$

At surface of sylinder, $v_r = 0$ and $r = a$

$$v_{\theta}/_{r=a} = -2u\sin\theta + \Gamma/2\Pi a$$

At stagnation points on cylindrical surface

$V_\theta = 0$:- $\sin\theta = \Gamma/4\Pi au$

When $\Gamma/4\Pi au < 1 \rightarrow$ we have two stagnation points.

When $\Gamma/4\Pi au = 1$ we have two stagnation points.

When $\Gamma/4\Pi au > 1 \rightarrow$ v_θ cannot be zero on the cylinder.

On the cylinder surface, $v_r = 0$

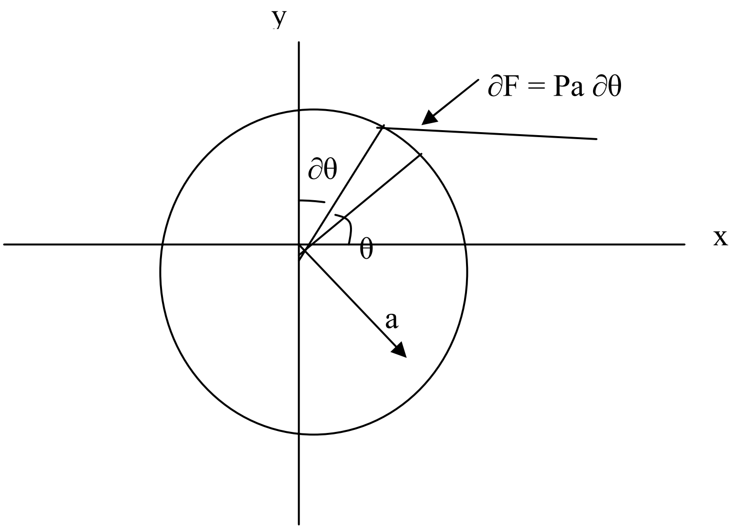
v_θ = total velocity

$p + \frac{1}{2} pv^2 = p + \frac{1}{2} p(v_\theta)^2{}_r = a = k$

$p = k - \frac{1}{2} p(4\, u^2 \sin^2\theta - \underline{2u\Gamma} \sin\theta + \Gamma^2/4\Pi^2 a^2)$

$p = k^1 - \frac{1}{2} p\, (4\, u^2 \sin^2\theta - 2ur/\Pi a \sin\theta)$

where $k1 = k - \frac{1}{2} \frac{p\Gamma}{4\Pi^2 a^2}$



$F_x = - \int_0^{2\Pi} pa \cos\theta \, d\theta$

$F_y = - \int_0^{2\Pi} pa \sin\theta \, d\theta$

Integration shown that $f_x = 0$

$Fy = - \frac{pu\Gamma}{\Pi}$ $F_y = - \int_0^{2\Pi} \sin^2\theta \, d\theta = pu\Gamma$

= Force per unit length acting on the cylinder.

Fy is perpendicular to direction of uniform rectilinear flow and is generally known as LIFT.

Fx distsparallel to the direction of uniform rectilinear flow is generally known as MAGNUS EFFECT. It was shown later by M.W. kutta and N.E Joukuwsli independently that for a body of any shape in 2 –D flow the transverse force per unit length is – puΓ in the plane of the flow and is perpendicular to the direction of flow. This result is known as kutta-Joukowski law and is one of the most useful results of ideal fluid flow theory.

Examples of Magnus effect.

(i) deflection of golf or tennis or cricket or ping pong balls which are "cut" or "sliced" or given a "top" spiri → transverse force. For a sphere, magnus effect is refers to as ROBINS EFFECT.

(ii) Flethner's rotor-ship which had large vertical cylinder on the deck.

(iii) Lift force on onrcraft wings blade of properties and turbores.

Elementary Aerofoil Theory

Commonly used terms in reference to aerofoils

Chord line: A straight line in the plane of the aerofoil cross-section, which serves as a datrum. It is commonly taken as the line joining the centres of curvature of the leading (i.e front) edge and trailing (i.e rear) edge.

Chord, c: The length of the chord line produced to meet the leading and trailing edges.

Span b: The overall length of the aerofoil (in the direction perpendicular to the cross section).

Plan Area, s: The area of the projection of the aerofoil on a plane perpendicular to the section (or profile) and containing the chord line. For an aerofoil with a cross section constant along the span, plan area = chord x span

Mean chord: $c = s/b$

Aspect ratio, AR or A = Span/mean chord $A = b/c = b^2/s$

Lift, L: That component of the total aerodynamic force on the aerofoil, which is perpendicular to the direction of the on winning fluid. Lift is not necessarily vertical.

<u>Drag, D</u>: That component of the total aerodynamic force on the aerofoil, which is parallel to the direction of the oncoming fluid.

<u>Lift Coefficient, CL</u>: L/(½ pu^2s)

Drag coefficient CD: D/(½ pu^2s) (U = velocity relative to the aerofoil of the fluid upstream).

<u>Angle of Attack, x</u>: The angle between the chord line and the direction of the oncoming fluid. More significantly, zero angle of attack is sometimes defined as that for which the A lift is zero.

<u>Example 5</u>

A source with strength $0.2m^2/s.m$ and a vortex with strength 1m2/s are located at the origin. Determine the equation for velocity potential and stream function. What are the velocity components at x = 1m and y = 0.5m7.

<u>Solution</u>:

The velocity potential for the source is

$\Psi = - \frac{0.2}{2\pi}$ enr m^2/s

And the corresponding stream function.

$\Psi = - \frac{q}{2\pi}$ ⍰ $= - \frac{0.2}{2\pi}$ ⍰ $m^2/>$

The velocity potential for the vortex is

$\Psi = - \frac{1}{2\pi}$ ⍰ m^2/s

And the corresponding stream function is

Ψ = - 1 enr m²/s

2π

Adding the respective functions gives

$\phi = -\frac{1}{\pi}(0.1\ \theta - \frac{1}{2}\ enr)$ and

$\Psi = -\frac{1}{\pi}(0.1\ \theta - \frac{1}{2}\ enr)$

The radial and tangential velocity components are

$V_r = -\frac{\partial \Psi}{\partial r} = \frac{1}{10\pi r}$

$V_\theta = -\frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{1}{2\pi r}$

At point (1, 0.5), $r = \sqrt{1^2 + 0.5^2}$ 1.11 7m/s

→ vr = 0.0285 m/s vθ = 0.143 m/s.

Example 6

Distinguish between free spiral and forced vortices give two examples of each.

Show that the horizontal variation of pressure in a free spiral vortex is given by $\frac{p_o - p}{pg} = \frac{c2 + (c/2\pi)^2\ 1/r2}{2g}$

Where c = strength of the free vortex

□ = source strength

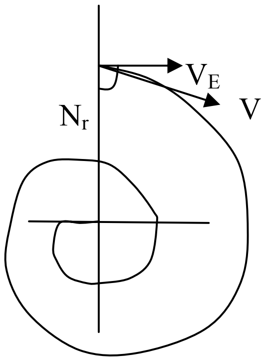
p_o = pressure as r → w.

Water leaner the guide passages of an inward-flow turbine at a radius of 1.2m. its velocity is than 2.0m/s at an angle of 70” to the radius. It enters the runner at a radius of 900mm. neglecting friction and assuming that the flow is entirely tow-dimensional, calculate the pressure drop between the guide passages and entry to the runner.

Solution:-

See pg . 32/45 x 27

See pg. 32



Vr = v sin□, vt = v cos□

Ri = 1.2m

V = 20m/s, □ = 70’

R2 = 900m = 0.9m

Vit = 20 cos 70’ = 6.84m/s

Vir = 20 sin 70’ = 18.794m

Vt = c/r

$C = rv1t = 1.2 \times 684$

$C = 8.208 \ m^2/s$

Also ⍰ $= 2\pi rvr$

$= 2\pi \times 1.2 \times 18.794$

$= 141.70$ m2/s

$$\frac{P_2 - p_1}{Pg} + (z_2 - z_1)^o = \frac{c^2 + (\psi/2\pi)^2}{2g} \{1/r^2{}_1 - 1/r^2{}_2\}$$

$$P_2 - p_1 = pg \left[\frac{c^2 + (\psi/2\pi)^2}{2g}\right] \{1/r^2{}_1 - 1/r^2{}_2\}$$

$$= 1000 \times 9.81 \times \left[\frac{8.108^2 + (141.7/2\pi)^2}{2 \times 9.81}\right] \times \{1/1.2^2 - 1/0.9^2\}$$

= - 155.55 kpa

:- $P_2 - p_1 = 155.6$ kpa

Example

A two-dimensional flow is described by the velocity components $u = 5x^3$ and $v = 15x^2y$. Determine the stream function, velocity and acceleration at point p (x = 1m, y = 2m).

Solution:

$U = 5x^3$, $v = -15x^2y$

$U = \frac{\partial \psi}{\partial y}$

$\Rightarrow \psi = \int u\, dy = \int 5x^3\, dy$

$= 5x^3y + f(x)$ ……………. (1)

$V = \frac{\partial \psi}{\partial x}$

$\Rightarrow \psi = \int -v dx = \int 5x^2 y dx$

$= 5x^3y + f(y)$ ……………. (2)

Equation (1) and (2)

$\Rightarrow \psi = 5x^3 y$

At point p (1,2)

$\Rightarrow \psi = 5x1^3x2$

$\psi = 10m2/5$

$\Rightarrow ux = 5x^3 = 5x1^3$

$= 5m/5$

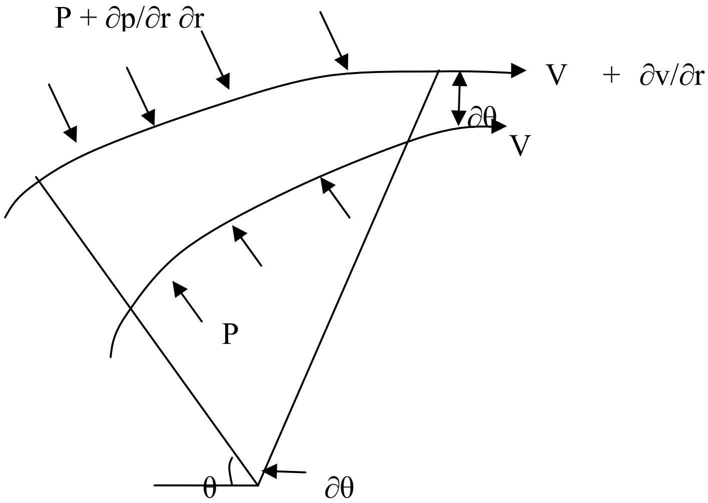
$\Rightarrow v = 15x2y$

$= 15x1^2x2$

V = - 30m/5

Vortices

Energy variation across curved stream lines



Example 7

A two-dimensivanl flow is described by the velocity components $u = 5x^3$ and $v = - 15x^2y$. Determine the stream function, velocity and acceleration at point p (x = 1m, y = 2m)

Solution:

$[U] = \sqrt{u^2 + v^2}$

$= \sqrt{5^2 + 30^2}$

[U] = 30.41m/d

$[a] = a^2_x + a^2_y$

$Ax = u \frac{\partial u}{\partial x} + \frac{v \partial u}{\partial y}$

$= (5x^3)(15x^2) + (-15x^2y)(0)$

= 75x5

$Ax = u \frac{\partial u}{\partial x} + \frac{v \partial u}{\partial y}$

$= (5x^3)(-30xy) + (-15x^2y)(-15x^2)$

$= -150x^4y + 225x^4y$

= - 300 + 450

$Ay = 150m/d^2$

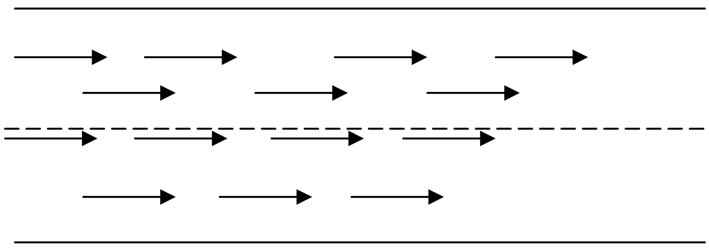
$\{a\} = \sqrt{75^2 + 150^2}$

[a] = 167 – 71m/s2

CHAPTER 3: FLOW IN PIPES

Laminar flow

In laminar flow, the fluid velocity is relatively low. The fluid particles move in uniform fasluion with their paths not crossing one another.



For laminar flow in pipes, the following expression gives the volume flow rate ?

$$? = \frac{t1\ R4\ dp}{8\ ml}$$

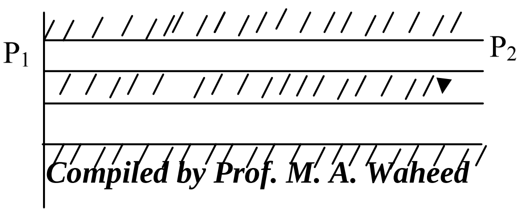
Where R = pipe inner radius

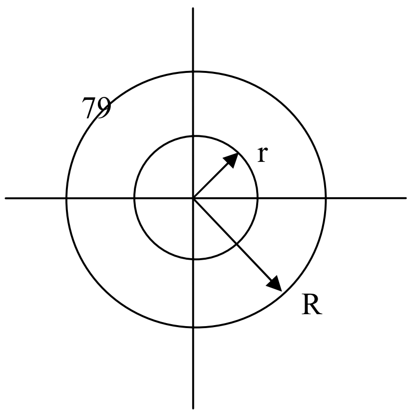
Dp = pressure difference between the 2 sections considered

N = Dynamic viscosity

L = length between the section

The expression is the poiseuille's formular a proof of which is given below.





$\Delta p = P_1 - P_2$

For the cylindrical pipe of radius r. the force it experience dive to the pressure difference is equal to the drag hence.

$$Dp.\ \pi r^2 = -\ u\ du/dr.\ \ \pi\ rl$$

In the expression for drag, the negative sign is present because du/dr (the velocity gradient) is negative.

At r = R, u = o, hence integration yield

$$U = \frac{dp\ (R^2 - r^2)}{4ml}$$

Using $\partial = \int^R_\partial u\ .\ 2\pi rdr$ and substituting for u we finally obtain.

$$Q = \frac{\pi R^4.\ Dp}{8ml}$$

Mean velocity $V = \underline{Q} = \underline{R2Dp}$

$\pi R^2 \quad 8NL$

Maximum velocity Umax = $\frac{Dp \,.\, R^2}{4NL}$ (Umax occurs arr = o)

i.e. Umax = 2V

Wall friction stress $\lambda\omega = - U\,(du/dr)_r = R$

From the expression of pressure force = Drag we obtain $\pi R^2\, dp = \lambda\omega.\, 2\pi rl$

$$\lambda\omega = \frac{RDP}{2l}$$

From the expression for mean velocity we have

$$Dp = \frac{V.8NL}{R^2}$$

$$\text{Hence } \lambda\omega = R/2L\, \left(\frac{V.\, 8ml}{R^2}\right)$$

$$\text{Or } \lambda\omega = \frac{4nv}{R}$$

A non-dimensional friction coefficient f is defined as

$$F = \frac{\lambda\omega}{\frac{1}{2}pv^2} = \frac{8nv}{pv^2R} = \frac{16n}{pvd} = \frac{16}{Re}$$

Furthermore, the Reynolds number the of flow through a pipe of diameter D is given by

$$Re = \frac{PVD}{N}$$

Reynolds number is a ratio of mertia to viewer's forces. When incrtial is small compared to viewer forces, the flow is lammar. When the reverse is the cade, the flow is tribulent. In calculating when Re < 2000, lammnar flow prevails.

Turbulent flow in pipes.

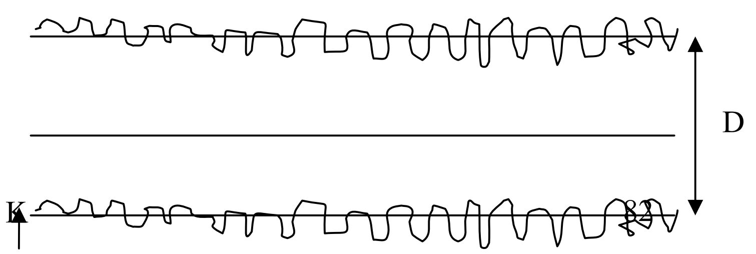
For this flow the velocity is relatively large fluid particles cross form layers to layers, i.e. the motion of a particular particle when observed in detail is zig-zag overall motion is parallel to pipe axis.

The turbulent flow situation occurs more commuuly in Engineering practice then the laminar flow situation. It has not be possible so far to derive entirely from first principle the expression which govern turbulent flow. A lot of experimentally derived equations are used. Generally it is taken that turbulent flow occurs in pipe when Re > 2500

Transition flow 2000 < Re < 2500

Lammnar flow Re < 2000

It is known that mechanical vibration encourage the onset of turbulent flow. Furthermore as the relative roughness of the pipe increases the tendency for turbulency to occur is high. Relative roughness is the average height of protrusion divided by the pipe diameter.

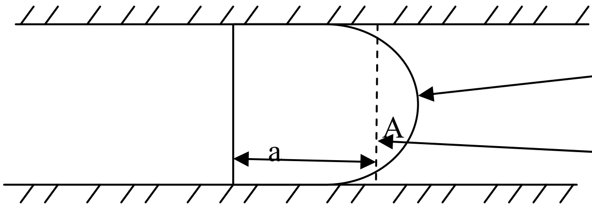


Microscopic view of a length of pipe

Relative roughness = K/D

V_E

New pipes are often taken as smooth and old one as rough.



Velocity gradient (and hence resistance) at the wall is greater for turbulent flow.

For turbulent flow, the ratio of mean velocity to maximum velocity is approximately 0.8 when Re is large. The resistance to flow increases with the mean velocity more rapidly than for lamer flow. This is because of eddies in the flow. When the flow is turbulent, there exists a laminar sub layer at transition zone and the fully turbulent zone. There are relations for the velocity distribution in each layer. These relations are not being presented here as they are applicable to boundary layer there. The following expression for friction factor has been obtained empirically and they are used when their respective condition holds.

For smooth pipes

F = 0.316 Re^0.25 for Re < 2 x 10^4

$F = 0.08\ Re^{-1/4,}$ Re < 80,000

Given by Blasins

Lamnar sub-layer higher than the protrusions.

Logarithmic resistance: formular from prandti $1/f = 4\ \log_{10}$ (Rejf) – 0.4

Re is as high as 3.4x106. Iterations are carried out with this equation. Note that f = f (Re) for smooth pipes.

Rough pipe

Here protrusions at the wall are higher than the laminar sub-layer. Alternatively it could be that Re is very large (up to 108, even if the pipe is new).

Nikuradsegare the relation

1/f = 4 log (D/2x) + 3.48

i.e f is independent of Reynolds number note that:

f = f (Re) = laminar flow

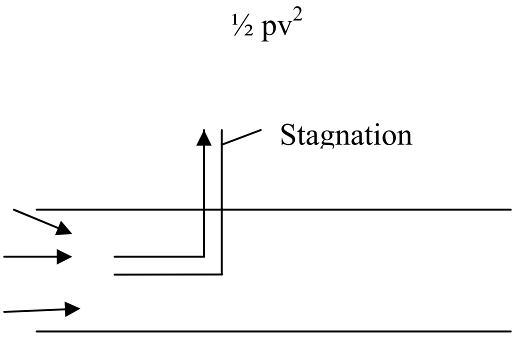
f = f (Re) – turbulent flow in smooth pipe

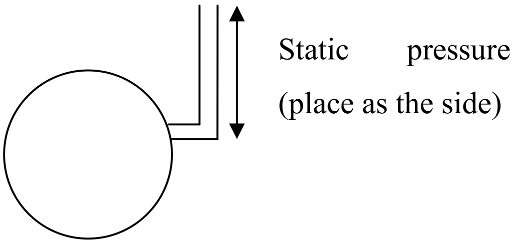
f = f(k/D) – turbulent flow in rough pipe.

<u>Friction factor f</u>

The friction factor is the ratio of wall shear stress to dynamic pressure

$f = \underline{\lambda\omega}$

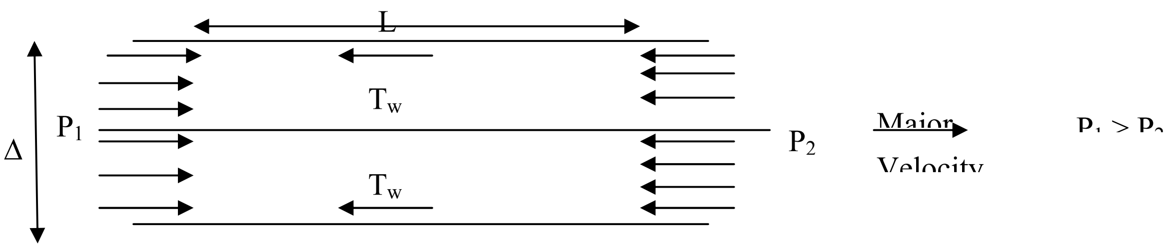




Dynamic pressure = stagnation pressure – static pressure = ½ pv^2

<u>Darcy – ileisbach equation</u>

This equation gives the frictional head loss hf for either turbulent or laminar flow in terms of friction factor, mean velocity, v, pipe length, L and pipe diameter.



$$V = \frac{pvd}{N} = \frac{900 \times 0.691\ 6 \times 75 \times 10^{-3}}{0.17}$$

Forces balance to give

$(p_1 - p_2)\, \pi D^2/4 = \lambda\omega\, \pi DL$

Using $p_1 - p_2 = hf\, pg$ and $\lambda\omega = f\, ½\, pv^2$ we can obtain

$$hf = \frac{4f\, v^2\, L}{2gD}$$

Example

An oil of specific gravity 0.9 and N = 0.17 kg/ms is pumped through a pipe whose diameter and length are 75mm and 750m respectively. The flow rate is 2.75 kg/s. check that the flow is larminar calculate the pressure drop in the pipe and the power required to overcome friction.

Solution:

$$Re = \frac{pvd}{N}$$

Volume flow rate ? $= \frac{\acute{m}}{P} = \frac{\pi D^2 V}{4}$

$$V = \frac{4\,\acute{m}}{p\pi D^2}$$

$$= \frac{4 \times 2.75}{}$$

$900 \times \pi \times (75 \times 10^{-3})^2$

$$V = \frac{pvd}{N} = \frac{900 \times 0.691\ 6 \times 75 \times 10^{-3}}{0.17}$$

Re = 274.6.

The flow is larminar

Forum poissenille's equation

$$Dp = \frac{8NL\ \Theta}{\pi R^4}$$

$$dp = \frac{8 \times 0.17 \times 750 \times 2.75}{\pi \times (37.5 \times 10^{-3})^4 \times 900}$$

Dp = 5.02 bar

Dp = hfpg

Hf = Dp/pg

Power = nighf

= pϴghf

= pϴg . Dp/pg

Power = ϴDp

= 1.53kw.

Water flows through a 50cm diameter pipe which may be regarded as rough. The flow rate is 0.5m3/s and the head loss per unit length is 0.05m. Taking m = 0.0013kg/ms for water, find the relative roughness of the pipe.

Solution:

ϴ = 0.5m3/s

$$V = \frac{4ϴ}{\pi D^2} = 2.546 m/s$$

$$hf = \frac{4f02L}{2gD}$$

$$\frac{hf}{L} = \frac{4fv2}{2gD} = 0.05$$

= f = 0.0189

$$Re = \frac{pud}{N} = \frac{1000 \times 1.5 \times 50 \times 10^{-1}}{1.3 \times 10^{-3}}$$

$= 9.792 \times 10^3$

Using charge

From f – Re – k/y charge

Relative roughness k/d = 0.057.

Questions:

At one time, water flows through a 25cm diameter pipe at the rate of $160 dm^3/hr$ and at another time at the rate of $680 dm^3/hr$.

The viscosity of water is 0.0013kg/ms. Using f = 16/Re for lammar flow and f = 0.064/Re for turbulent flow, compare the frictional losses for the two conditions of flow.

An oil water consists of 200 tubes with each tube having an internal diameter of 12mm and a length of 3.5m. An oil of specific gravity 0.9 is forced through the tubes at a speed of 1.8m/s. The viscosity of the oil varies linearly from the inlet to the outlet. At the inlet the viscosity is 0.029kg/ms while at the outlet it is 0.1kg/ms. Calculate the power required to pump the oil through the cooker. (3.65kg).

<u>Pipe losses</u>

Darcy-weisbach equation gives the lose along the length of the pipe. Losses also occur in pipe fittings such as expansions, contractions, elbows, bends, valves, etc.

These losses are minor and may be neglected only when the pipe length is short, losses in fittings may be major. The losses invariably arise from sudden changes of velocity which generate large-scale turbulence in which energy is dissipated as heat.

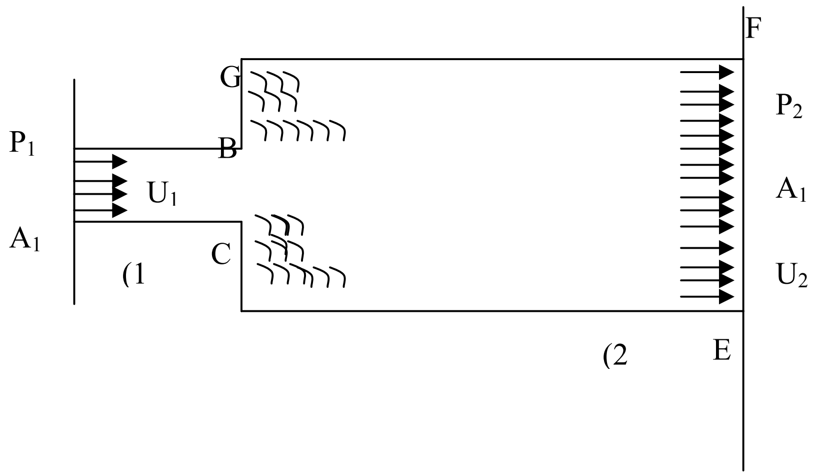
The source of the loss is usually unfired to a very short length of the pipe, but turbulence produced may persist for a considerable distance downstream. The total head lost in a pipe may be calculated as the sum of the normal friction for the length of pipe considered and the additional bosses.

The losses in fittings are frequently expressed in the form.

$$\text{Head lose} = k\, u^2/2g$$

The value of k is practically constant at high Reynolds number.

Loss at abrupt enlargement



Consider flow in a pipe with sudden enlargement as shown above. Fluid emerging from the smaller pipe is unable to follow the abrupt deviation of the boundary, resulting in pockets of turbulent eddies formation in the corners and dissipation of energy as heat.

Velocity u_1 and u_2 are uniform with $u_2 < u_1$.

The net force acting towards the right of ht control volume BCDEFG is

$$P_1A_1 + p^1 (A_2 - A_1) - p_2A_2$$

Where pi represents the mean pressure of the eddying fluid over the annular face GD and is sensibly equal to p1. The net force on the fluid equals the rate of increase of momentum in that direction.

$$(P_1 - p_2)A_2 = pQ (u_2 - u_1)$$

$$P_1 - p_2 = \frac{pQ}{A_2} (u_2 - u_1) = pu_2 (u_2 - u_1)$$

From the energy equation for a constant density fluid we have.

$$P1/pg + u2/2g + z_1 = p^2{}_1/pg + u^2{}_2/2g + z_2 + hf$$

:- $$hf = \frac{p_1 - p_2}{Pg} + \frac{u^2{}_1 - u^2{}_2}{2g}$$

Substituting the expression for $p_1 - p_2$

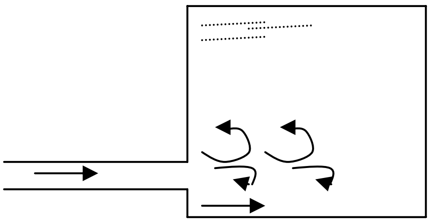
=> $$hf = u_2 (u_2 - u_1) + u^2{}_1 - u^2{}_2 = (u_1 - u_2)^2$$

Since A1 u1 = A2u2

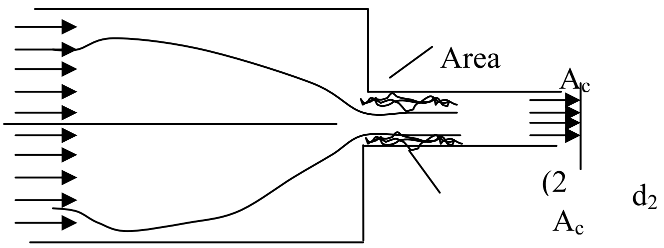
$$\Rightarrow \qquad hf = \frac{u_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 = \frac{u_2^2}{2g}\left(\frac{A_2}{A_1} - 1\right)^2$$

Exit loss

If $A_2 \rightarrow$ p1 the head loss at an abrupt enlargement tends to u21/2g. This occurs at submersed outlet of a pipe discharging into a large reservoir. The loss is usually termed the exit loss for the pipe.



Loss at sudden contraction, elbours, etc.



U_2

A_2

In this flow, a vena contractor is formed immediately downstream of the junction. The flow after the vena contracta downstream section (1) is uniform. The lost of head is assumed to be given by

$$hf = \frac{u_2^2}{2g}\left(\frac{A_2}{A_2} - 1\right)^2 = \frac{u_2^2}{2g}\left(\frac{1}{\varepsilon} - 1\right)^2$$

Where Ac represents the cross-selfonal area of the vena contracta and the coefficient of contraction, $C_c = A_c/A_2$. The value of C_c depends explicityly on A_2/A_1. The lost of head can be determined using the following expression.

$$Hp = k\,u^2/2g$$

Where the values of k is tabulated as a friction of d_2/d_1

d2/d1	0	0.2	0.4	0.6	0.8	1.0
K	0.5	0.45	0.38	0.28	0.14	0.00