Course Code: MTS 423

Course Title: Functional Analysis

Number of Units: 3 Units

Course Duration: Three hours per week

COURSE DETAILS:

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COURSE CONTENT:

1. Metric Spaces Review (Baries Cateogies Theorem)
2. Banach Spaces (Definition, Examples and elementary theory)
3. Hilbert Spaces (Definition, Examples and elementary theory)
4. Operators on Banach/Hilbert Spaces
5. Fundamental Theorems of Functional Analysis (Open mapping, Closed graph , Hahn Banach Theorem and Uniform boundedness principle)

COURSE REQUIREMENTS:

This is a compulsory course for all students in the Department of Mathematics. The knowledge from MTS223, MTS212, MTS323 and MTS362 are required.It is also expected that students participate in all the course activities and have a minimum of 75 percent attendance to be able to qualify for the final examination. The grading will be based on weekly homework assignment (10 percent), an in class - mid term test (20 percent) and a final examination (70 percent).

READING LIST:

The following are recommended:

1. C. Goffman and G. Pedrick; First course in functional analysis - Chelsea (1983).

2. J.B. Conway;, A course in functional analysis, Springer Verlag (1985).

3. C.E. Chidume; Applicable functional analysis.

4. N.I. Young; An introduction to Hilbert space, Cambridge University Press (1988).

5. W. Rudin; Functional analysis, McGraw - Hill (1991).

Brief Introduction:

Functional analysis deals with objects which have both an algebraic structure (such as vector space or an algebra) and an analytic structure (such as norm or topology) and for which the algebraic and analytic structure are connected in some way. For example, Banach space, in which the algebraic structure comes from the vector space and the analytic structure comes from the norm. Another interesting example is the Banach algebra, in which the algebraic structure comes from the algebra and the analytic structure comes from the algebra norm.

In this course, we shall learn two of the important spaces in functional analysis (Banach and Hilbert spaces) with examples. We shall also discuss some general theory and operators defined on them. The four fundamental theorems in functional analysis (Open mapping, closed graph, Hahn Banach theorems and uniform boundedness principle) will be introduced and studied at elementary level.

1 Metric Spaces Review (Baries Cateogies Theorem)

We briefly review the definition and some basic concepts on Metric spaces. For detail, see MTS 362 lecture note.

Definition 1.1 *Let X be any non-empty set, a metric on X is a mapping $d : X \times X \to \Re$ which satisfies the following consitions:*
(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0$ if and only if $x = y$
(iii) $d(x, y) = d(y, x)$ (symmetric property)
(iv) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)
for every $x, y, z \in X$.
The pair (X, d) is called a metric space.

Remark: If d satisfies (i), (iii) and (iv), then we shall call (X, d) a semi-metric space or pseudo metric space.

Definition 1.2 *The product of two metric spaces (X_1, d_1) and (X_2, d_2) is the space*

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

with the metric

$$d((a_1,a_2),(b_1,b_2)) = max\{d_1(a_1,b_1), d_2(a_2,b_2)\}.$$

Two alternative metrics on $X_1 \times X_2$ are the "taxi-cab metric"

$$d'((a_1,a_2),(b_1,b_2)) = d_1(a_1,b_1) + d_2(a_2,b_2)$$

and

$$d''((a_1,a_2),(b_1,b_2)) = \sqrt{d_1(a_1,b_1)^2 + d_2(a_2,b_2)^2}$$

Assignment 1

1. Show that d, d', d'' are metric on $X_1 \times X_2$ and that for $(a,b) \in X_1 \times X_2$,

$$d(a,b) \leq d''(a,b) \leq d'(a,b) \leq 2d(a,b).$$

2. Let (X,d) be a metric space. Show thar d_1 defined by

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is also a metric on X.

3. Show that if $K_1 \subset X_1$ and $K_2 \subset X_2$ are compact subsets of metric spaces X_1, X_2, then $K_1 \times K_2$ is a compact subset of the metric space $X_1 \times X_2$.

Definition 1.3 *A subset A of a metric (X,d) is said to be nowhere dense in X if its closure $\overline{A}$ in X does not contain a non-empty subset of X.*

Definition 1.4 *A subset A of a metric (X,d) is said to be of first category in X if A can be expressed as the union of a finite/countable family of nowhere dense sets. Otherewise A is of the second category.*

Theorem 1.5 *(Bairies Category Theorem) A non-empty complete metric space is of the second category.*

Proof To be provided in class.

2 Banach Spaces (Definition, Examples and elementary theory)

In this section, the knowledge of vector space and some basic results on vector space is needed. This was covered in MTS212. For detail, see your MTS212 lecture note.

We begin with the definition of normed linear space, with examples.

Definition 2.1 *Let X be a vector space over a field $\mathcal{F}$, where $\mathcal{F} = \Re$ or $\mathbb{C}$. A norm on X is a real valued function $\|.\|$, $\|.\| : X \to \Re$ such that the following conditions are satisfied:*
(i) $\|x\| \geq 0$
(ii) $\|x\| = 0$ *if and only if* $x = 0$
(iii) $\|\alpha x\| = |\alpha|\|x\|$ *for all* $\alpha \in \mathcal{F}, x \in X$
(iv) $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in X$.

A vector space X with a norm $\|.\|$ on it, that is, the pair $(X, \|x\|)$ is called a normed linear space (or normed space).

Remark:
When $\mathcal{F} = \Re$, we have a real normed linear space and when $\mathcal{F} = \mathbb{C}$, we have a complex normed linear space. Throughout this note, except when stated otherwise, we shall assume that the base field is $\mathbb{C}$ which is more useful case,

but we need to note that the real theory is the same except where otherwise noted.

Proposition 2.2 *Let $(X, \|x\|)$ be a normed linear space. Then*

$$d(x, y) = \|x - y\| \quad (x, y \in X)$$

defines a metric on X.

Proof To be provided in class.

Remark:

The metric d defined above is called the "associated metric". It will simply be assumed that if we are working in a normed linear space terms such as convergent sequence, open set, closed set, continuous function, are to be understood in the sense of this metric. For example, a sequence (x_n) converge to $x \in X$ if and only if $\|x_n - x\| \to 0$ and a function $f : X \to \mathbb{C}$ is continuous if and only if for every $x \in X$ and $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\|x - y\| < \delta$, we have $\|f(x) - f(y)\| < \epsilon$.

Example 2.3

1. *The scalar fields $\Re$ and $\mathbb{C}$ are normed linear spaces with absolute value defined as norm on them.*

2. *Let $X = \Re^2$. For arbitrary $\overline{x} = (x_1, x_2) \in X$, define $\|.\|_2 : \Re^2 \to \Re^+$ by*

$$\|\overline{x}\|_2 = (x_1^2 + x_2^2)^{\frac{1}{2}}.$$

Then $\|.\|_2$ is a norm on $\Re^2$.

3. *For $n \in \mathbb{N}$, the space $\mathbb{C}^n$ with the norm*

$$\|(x_1, x_2, ..., x_n)\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$$

is a normed linear space.

4. *For* $n \in \mathbb{N}$ *and* $1 \leq p < \infty$, *the space* $\mathbb{C}^n$ *with the norm*

$$\|(x_1, x_2, ..., x_n)\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$$

are normed linear spaces.

5. *For* $n \in \mathbb{N}$, *the space* $\mathbb{C}^n$ *with the norm*

$$\|(x_1, x_2, ..., x_n)\|_\infty = max_{1 \leq i \leq n} |x_i|$$

is a normed linear space. It can be shown that $\|\overline{x}\|_p \rightarrow \|\overline{x}\|_\infty$ *as* $p \rightarrow \infty$ *for all* $\overline{x} \in \mathbb{C}^n$.

6. *For* $1 \leq p < \infty$, *the spaces* l^p *defined by*

$$l^p = \{\overline{x} = (x_1, x_2, ...), x_i \in \mathcal{F} = \Re \ \ or \ \ \mathbb{C} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

with the norm

$$\|\overline{x}\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$$

for $\overline{x} = (x_1, x_2, ...) \in l^p$ *are normed linear spaces.*

7. *The space* l_∞ *defined by*

$$l_\infty = \{\overline{x} = (x_1, x_2, ...), x_i \in \Re : \overline{x} \ \ is \ \ bounded\}$$

with the norm

$$\|\overline{x}\|_\infty = sup_{i \geq 1} |x_i|$$

for $\overline{x} \in l_\infty$ *is a normed linear space.*

8. *The space c of convergent sequences with the norm*

$$\|\overline{x}\|_\infty = sup_{i\geq 1}|x_i|$$

is a normed linear space.

9. *The space c_0 of sequences converging to zero with the norm*

$$\|\overline{x}\|_\infty = sup_{i\geq 1}|x_i|$$

is a normed linear space.

Remark:

c and c_0 are proper subspaces of l_∞.

Proposition 2.4 *If $p < q$, then $l^p \subset l^q$.*

Proof Exercise

Theorem 2.5 *(Continuity of algebraic operations) Let $(X, \|.\|)$ be a normed linear space. Then*

1. *the map $x \to \|x\| : X \to \Re^+$ is continuous*

2. *if $x_n \to x$ and $y_n \to y$ in X, then $x_n + y_n \to x + y$*

3. *if $x_n \to x$ in X and $\lambda_n \to \lambda$ in $\mathbb{C}$, then $\lambda_n x_n \to \lambda x$.*

Proof To be provided in class.

Assignment 2

1. Let $a, b > 0$ and $X = \mathbb{C}^2$. Show that the function

$$\|\overline{x}\| = a|x_1| + b|x_2| \quad (\overline{x} = (x_1, x_2)$$

is a norm on X.

2. Let $(X, \|.\|)$ be a normed linear space. Define a function $\|.\|'$ on X by

$$\|x\|' = \frac{\|x\|}{1 + \|x\|} \quad (x \in X).$$

Prove or disprove that $\|.\|'$ is a norm on X.

3. Let $(X, \|.\|)$ be a normed linear space.

(a) For $k > 0$, prove that the set

$$X_k = \{x \in X : \|x\| \leq k\}$$

is convex.

(b) Let C be a convex subset of X, prove that its closure $\overline{C}$ is also convex.

4. Let X and Y be two normed linear spaces. For $(x, y) \in X \times Y$, define

$$\|(x, y)\| = \|x\| + \|y\|.$$

Show that $X \times Y$ is a normed linear space with this norm.

5. Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q} \quad (x, y > 0).$$

(b) (Holder's Inequality): For $x = (x_1, x_2, ...), y = (y_1, y_2, ...)$, then

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

Which known inequality do we obtain for $p = q = 2$?

(c) (Minkowski's Inequality): For $x = (x_1, x_2, ...), y = (y_1, y_2, ...)$, then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

6. Let $X = \prod_i^n X_i$, where each X_i is a normed linear space. Show that X is a normed linear space.

Definition 2.6 *A normed linear space is called a Banach space if it is complete (i.e, if every Cauchy sequence in it converges to a point it).*

Completeness is a very important concept in functional analysis and this will become more evident in subsequent sections.

To check or verify that a normed linear space X is complete, we take an arbitrary Cauchy sequnce in it and show that it converges to a point in it. The general pattern is the following:

1. Construct an element x^* which is to be used as the limit of the Cauchy sequence
2. Prove that X^* is in the space under consideration
3. Prove that $x_n \to x^*$ (in the sense of the norm or metric under consideration).

Example 2.7 *1. $\Re$ and $\mathbb{C}$ are complete.*

2. $(\Re^n, \|.\|_p$ where

$$\|\overline{x}\|_p^n = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$$

is complete.

3. *For* $1 \leq p < \infty, (l^p, \|.\|_p)$ *is complete.*

4. *The space* $C[a,b]$ *of continuous real valued functions on* $[a,b]$ *with the sup norm is complete.*

Some important normed linear spaces are not complete.

Example 2.8 *Let* $X = C[a,b]$ *with the norm given by*

$$\|f\| = \int_a^b |f(t)|dt \quad (t \in [a,b], f \in C[a,b]$$

is not complete.

Assigment

1. let $(X, \|.\|_X$ and $(Y, \|.\|_Y$ be two normed spaces. For $(x,y) \in X \times Y$, define

   $$\|(x,y)\| = \|x\|_X + \|y\|_Y.$$

   (a) Show that a sequence (x_n, y_n) converges to $(x,y) \in X \times Y$ if and only if (x_n) converges to $x \in X$ and (y_n) converges to $y \in Y$.

   (b) Show that if X and Y are complete, so is $X \times Y$.

2. Let Y be a subspace of a Banach space X. Show that Y is complete if and only if it is a closed subspace of X.

3. Show that the space $C[-1,1]$ with the norm

   $$\|f\| = \int_{-1}^1 |f(t)|dt \quad (t \in [a,b], f \in C[-1,1]$$

   is not complete.

3 Hilbert Spaces (Definition, Examples and elementary theory)

In this section, we introduce a special class of Banach space called the Hilbert space.

Definition 3.1 *Let x be a vector space. An inner product also called a dot product or scalar product on X is a mapping $\langle .,\rangle : X \times X \to \mathcal{F}(\mathcal{F} = \Re$ or $\mathbb{C})$ which satisfies the following conditions:*

1. $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0$ *if and only if* $x = 0$
2. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
3. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$
   for each $x, y, z \in X, \alpha, \beta \in \mathcal{F}$.

The pair $(X, \langle ., .\rangle)$ is called an inner product space (IPS).

Remark

1. For $x \in X$, we define $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. Hence, every IPS is a normed linear space and hence a metric space.
2. From (2) and (3) of definition 3.1, we have

$$\langle z, \alpha x + \beta y\rangle = \overline{\alpha}\langle z, x\rangle + \overline{\beta}\langle z, y\rangle$$

   for each $x, y, z \in X, \alpha, \beta \in \mathbb{C}$.

Proposition 3.2 *Let X be an IPS, then for $x \in X$, $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ is a norm on X.*

Definition 3.3 *Let $(X, \langle ., . \rangle)$ be an IPS, then $(X, \langle ., . \rangle)$ is called an Hilbert space if $(X, \|.\|)$ is complete where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. That is, a complete IPS is called a Hilbert spacee*

We next give some examples.

Example 3.4 *1. $\mathbb{C}^n$ with $\langle a, b \rangle = \sum_{i=1}^{n} a_i \overline{b_i} \quad (a, b \in \mathbb{C}^n)$ is an IPS. item $C[a, b]$ with $\langle f, g \rangle = \int_a^b x(t)\overline{y(t)}dt$ is an IPS.*

2. l_2^n with $\langle a, b \rangle = \sum_{i=1}^{n} a_i \overline{b_i} \quad (a, b \in l_2^n)$ is an IPS.

We next give some basic properties of IPS.

Proposition 3.5 *(Cauchy-Schwartz Inequality) Let $(X, \langle ., . \rangle)$ be an IPS. For any two vectors $x, y \in X$, then*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle.$$

Equality holds if and only if x and y are linearly dependent.

Proof To be provided in class

Proposition 3.6 *(Parallelogram law) Let $(X, \langle ., . \rangle)$ be an IPS. Then for any $x, y \in X$,*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof To be provided in class.

Proposition 3.7 *(Polarization identity) Let $(X, \langle ., . \rangle)$ be an IPS. Then for any $x, y \in X$,*

$$\langle x, y \rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right) \quad (i^2 = -1).$$

Proof To be provided in class.

Definition 3.8 *The vectors x, y in an IPS X are said to be orthogonal if $\langle x, y\rangle = 0$ and this is denoted by $x \perp y$.*

If A is a subset of X, then we write $x \perp A$ if $x \perp y$ for every $y \in A$.

Definition 3.9 *A set S in an IPS X is called an orthogonal set if $\langle x, y\rangle = 0$, for each $x, y \in S, x \neq y$. The set S is called an orthonormal set if it is an orthogonal set and $\|x\| = 1$ for each $x \in S$.*

Theorem 3.10 *Let A be a non-empty set in an IPS X. Then the set of all vectors orthogonal to every vector in A is a closed subspace of X. This subspace is called the orthogonal complement of A and it is denoted by $A^{\perp}$.*

Proof To be provided in class.

Theorem 3.11 *An orthonormal set of vectors in an IPS X is linearly independent.*

Proof To be provided in class.

Definition 3.12 *Let X be a vector space. X is said to be the direct sum of two subspaces M and N of X written as $X = M \oplus N$ if each $x \in X$ can be represented uniquely as $x = m + n$ with $m \in M, n \in N$. In this case, N is called the algebraic complement of M in X (and vice versa). The subspaces M and N are called complementary pair of subspaces in X.*

Proposition 3.13 *Let M and N be arbitrary subspaces of a Hilbert space H. Then*

1. $M^\perp$ *is a closed subspace of* H

2. $M \subset M^{\perp\perp}$

3. *If* $M \subset N$, *then* $N^\perp \subset M^\perp$

4. $M^{\perp\perp\perp} = M^\perp$

Proof To be provided in class.

Theorem 3.14 *Let* M *be a closed subspace of a Hilbert space* H, *then* $H = M \oplus M^\perp$.

Proof To be provided in class.

4 Operators on Banach/Hilbert Spaces

In this section, the knowledge of linear maps defined on vector spaces and some basic concepts/results on linear maps are needed. These were covered in MTS212. For detail, see your MTS212 lecture note.

Definition 4.1 *Let* X *and* Y *be normed linear spaces over a scalar field* $\mathcal{F}$, *and let* $T : X \to Y$ *be a linear map. Then* T *is said to be bounded if there exists some constant* $M > 0$, *such that* $\|T(x)\| \leq M\|x\|$ *for each* $x \in X$.

We now focus on linear maps that are continuous. The concept of continuity for linear maps can be stated in several equivalent forms. We state these equivalent forms in the next theorem.

Theorem 4.2 *Let X and Y be normed linear spaces over a scalar field $\mathcal{F}$, and let $T : X \to Y$ be a linear map. Then the following statements are equivalent:*

1. *T is continuous*

2. *T is continuous at the origin (in the sense that if (x_n) is a sequence in X such that $X_n \to 0$ as $n \to \infty$, Then $T(x_n) \to 0$ in Y as $n \to \infty$.*

3. *T is Lipschitz i.e there esists a constant $K > 0$, such that for each $x \in X$, $\|T(x)\| \leq K\|x\|$.*

4. *If $D = \{x \in X : \|x\| \leq 1\}$ is the bounded unit disc in X, then $T(D)$ is bounded (in the sense that there exists a constant $M > 0$ such that $\|T(x)\| \leq M$ for every $x \in D$.)*

Proof To be provided in class.

Remark The above result shows that for linear maps, continuity and boundedness are equivalent.

Example 4.3 *Let X and Y be normed linear spaces and let $B(X,Y)$ denotes the family of all bounded linear maps from X into Y. Then $B(X,Y)$ is a vector space with addition and scalar multiplication defined by*

$$(T + L)(x) = T(x) + L(x)$$

$$(\alpha T)(x) = \alpha T(x) \quad (T, L \in B(X,Y), \alpha \in \mathcal{F}).$$

Also, $B(X,Y)$ is normed linear space with the norm defined by

$$\|T\| = sup_{\|x\| \leq 1} \|T(x)\|.$$

Assignment *Let X and Y be normed linear spaces. Show that $B(X, Y)$ with norm defined in Example 4.3 is complete if and only if Y is complete.*

Since $\Re$ and $\mathbb{C}$ are complete, then if $Y = \Re$ or $\mathbb{C}$, then $B(X, \Re)$ is complete, that is a Banach space with the norm defined by

$$\|f\| = sup_{\|x\| \leq 1} |f(x)| \quad (f \in X').$$

This is denoted by X' and it is called the dual space of X. That is $X' = B(X, \Re)$.

Example 4.4

1. *The dual space of c_0 is l^1.*
2. *The dual space of l^1 is l^∞.*
3. *For $1 < p < \infty, l^{p'} = l^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Definition 4.5 *The bidual X'' of a normed linear space X is simple the dual of the dual space X' of X. That is $X'' = (X')'$.*

The key fact about bidual is the following:
There exists a canonical injection $v : X \to X''$ defined for each $x \in X$ by $v(x) = \varphi_x$, where $\varphi_x : X' \to \Re$ is given by $\varphi_x(f) = \langle f, x \rangle \quad (x \in X, f \in X')$. Thus, $\langle v(x), f \rangle = \langle f, x \rangle \quad (x \in X, f \in X')$.

Note: By notation, $\langle f, x \rangle = f(x)$.

Assignment*:*

1. *Show that the map v defined above satisfies the following:*

   (a) v is linear

(b) v is an isometry i.e $\|v(x)\| = \|x\|$ for all $x \in X$.

2. *Let x be an element in a normed linear space X. Show that*

$$\|x\| = sup\{|f(x)| : f \in X', \|f\| = 1\}.$$

In general, the map v defined above need not be onto. Consequently, we alwats identity X as a subspace of X''. Since an isometry is always injective, it follows that v is an isomorphism onto $v(X) \subset X''$. This leads to the next definition.

Definition 4.6 *Let X be a normed space and let v be the canonical embedding of X into X''. If v is onto, then X is called reflexive. Thus, a reflexive Banach space is one in which the canonical embedding is onto.*

Note: v is onto implies $v(X) = X''$ and in this case, we write $X = X''$ to mean X is reflexive.

lastly in this section, we discuss operators defined on Hilbert spaces.

Definition 4.7 *Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator. Then the adjoint T^* of T is a map $T^* : H \to H$ defined by*

$$\langle Tx, y\rangle = \langle x, T^*\rangle \quad (x, y \in H).$$

Remark*:*

1. *T^* always exist*
2. *T^* is unique*
3. *T^* is linear and bounded*

We next give some basic properties of the adjoint operator T^* of T.

Theorem 4.8 *Let T, T_1 and T_2 be bounded linear operators on a Hilbert space H into itself. Then the adjoint T^* of T has the following properties:*

1. $I^* = I$, *where I is the identity operator on H.*
2. $(T_1 + T_2)^* = T_1^* + T_2^*$
3. $(\alpha T)^* = \overline{\alpha} T^*$
4. $(T_1 T_2)^* = T_2^* T_1^*$
5. $T^{**} = T$
6. $\|T^*\| = \|T\|$
7. $\|T^* T\| = \|T\|^2$
8. *If T is invertible so is T^* and $(T^*)^{-1} = (T^{-1})^*$.*

Proof *To be provided in class.*

Definition 4.9 *Let H be a Hilbert space. An operator $T : H \to H$ is called;*

1. *Self adjoint or Hermitian if $T = T^*$*
2. *Normal if $T^* T = T T^*$*
3. *Unitary if $T^* T = T T^* = I$*

Theorem 4.10 *The self-adjoint operators on a Hilbert space H form a closed, real linear subspace of $B(H)$.*

Proof *To be provided in class.*

Theorem 4.11 *Let T_1 and T_2 be self adjoint operators on a Hilbert space H. Then T_1T_2 is self adjoint if and only if $T_1t_2 = T_2T_1$.*

Proof *To be provided in class.*

Theorem 4.12 *An operator T on a Hilbert space H is self adjoint if and only if $\langle Tx, x\rangle$ is real for every $x \in X$.*

Proof *To be provided in class.*

Theorem 4.13 *The set of all normal operators on a Hilbert space H is a closed subspace of $B(H)$ which contains the set of all self-adjoint operators and it is closed under scalar multiplication.*

Proof *To be provided in class.*

Remark*:*

Unitary operators are normal. They are the non-singular operators whose inverses equal their adjoint.

Assignment

1. *Let T be an operator on a Hilbert space H, prove that the following statements are equivalent:*

   (a) $T^*T = I$

   (b) $\langle Tx, Ty\rangle = \langle x, y\rangle$

   (c) $VertTx\| = \|x\|$ *for all* $x \in X$

2. *Let $T : H \to H$ be a bounded linear operator where H is a Hilbert space. Suppose T can be written in the form*

$$T = \frac{1}{2}(T + T^*) + i[\frac{1}{2i}(T - T^*)] = A + iB.$$

Show that

(a) *A and B are self adjoint*

(b) *T is normal if and only if $AB = BA$.*