UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA

DEPARTMENT OF MATHEMATICS

Course Code	**MTS 461**
Course Title	**GENERAL TOPOLOGY**
Number of Units	**3 units**
Course Duration	**3 Hours per week**

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Course Outline:

Topological Spaces:Definition and Examples, Neighbourhood and Neighbourhood systems, Subspaces, Induced Topology. Bases and subbases, Continuity in Topological Spaces, First and second countable spaces, Separation axioms : T_1, T_2, T_3, T_4 spaces, Hausdorff, Regular, Normal spaces, Compactness, Connectedness.

Prerequsite: MTS 362 - Metric Spaces

Textbooks

1. James R. Munkres; Topology, 2nd Edition, Prentice Hall Inc., USA, 2000.

2. James Dugundji, Wm. C. Brown Publishers, Dubuque, IOWA, USA, 1989.

3. G. F. Simmons, Introduction to Topology and Modern Analysis, 2nd Edition, McGraw Hill, New York.

4. Bashir Ahmad, Introduction to General Topology, 2nd Edition, Idea Publishers, Multan, 2004.

5. Iain Adamson, A General Topology Workbook, Birkhäuser Publishers, Boston, 1996.

6. P. K. Geetha – Topics in Moderm Mathematics, Matscience Report 69, Institute of Mathematical Sciences, Madras-20, India.

7. J. L. Kelley; General Topology, Springer-Verlag, New York, 1991.

8. Plus any standard text in topology.

What is expected of the Student:

Students are expected to attend all lectures and complete all quizzes, assignments and examinations. No aids are permitted in quizzes and examinations.

Evaluation of Student Performance:

1. Midsemester Examination: 15% (Date and lenght to be determined).

2. Written Assignments: 10% (Dates to be announced).

3. Quizzes: 5% (Dates not to be specified).

4. Final Examination: 70% ($2\frac{1}{2}$hrs, date to be determined and fixed by TIMTEC).

1 Topological Spaces:Definition and Examples

Definition 1.1 *Let X be a non-empty set. A family (class) $\mathcal{T}$ of subsets is called a topology on X if i and only if it satisfies the following axioms:*

(i) *X, $\emptyset$ belong to $\mathcal{T}$, where $\emptyset$ is the empty set*

(ii) *the arbitrary union of sets in $\mathcal{T}$ belongs to $\mathcal{T}$ (arbitrary union)*

(iii) *the intersection of any two sets in $\mathcal{T}$ (hence finite sets) belongs to $\mathcal{T}$ (finite intersection).*

The sets belonging to $\mathcal{T}$ are called $\mathcal{T}$ -open sets and the pair $(X, \mathcal{T})$ is called a *topological space.* When the underlying topology is understood, we simply speak of 'open sets' relative to that topology and usually denote the corresponding topological space by X.

Example 1.2 *Let X be a non-empty set.*

(i) *If $\mathcal{T}$ $=\{X, \emptyset\}$, then $\mathcal{T}$ is called the trivial or indiscrete topology and the corresponding topological space is called the indiscrete space.*

(ii) *If $\mathcal{T}$ is the class of all subsets of X, then $\mathcal{T}$ is called the discrete topology and X, together with $\mathcal{T}$, is called the discrete space.*

(iii) *Let $X = \mathbb{R}$ and $\mathcal{T}$, the class of unions of open intervals on $\mathbb{R}$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$, called the usual topology on $\mathbb{R}$. Similarly, the two-dimensional space $\mathbb{R}^2$, together with the topology constituted by all the open discs is another topological space.*

(iv) *Consider $X = \{a, b, c, d, e\}$. Define*

$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b, c\}\}$

$\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{a, c, d\}, \{b, c, d, e\}\}$

$\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\} \{a, b, c, d\}\}$

$\mathcal{T}_1$ is not a topology, since $\{a\} \cup \{b\} = \{a, b \notin \mathcal{T}_1\}$, while $\mathcal{T}_2$ fails to be a topology as $\{a, c, d\} \cap \{b, c, d, e\} = \{c, d\} \notin \mathcal{T}_2$. However, $\mathcal{T}_3$ is a topology (verify).

Example 1.3 *Let X be a non-empty set.*

(i) If $\mathcal{T}$ consists of $\emptyset$ and all those subsets of X whose complements are finite, then $\mathcal{T}$ is a topology and it is known as a co-finite topology on X (verify).

(ii) If $\mathcal{T}$ consists of $\emptyset$ and all those subsets of X whose complements are countable, then $\mathcal{T}$ is a topology on X called the co-countable topology (verify).

Example 1.4 *Let $X = \mathbb{R}$ and define $\mathcal{T}$ to be the class of unions of open-closed intervals $(a, b]$. Then $\mathcal{T}$ forms a topology on $\mathbb{R}$, called the upper limit topology on $\mathbb{R}$. Similarly, a class of unions of closed-open intervals $[a, b)$ forms a topology, called the lower limit topology on $\mathbb{R}$.*

Example 1.5 *Let (X, d) be a metric space and let the topology be the class of opoen sets of this metric space. Then such a topology is called a metric topology generated by the metric d and is denoted by $\mathcal{T}_d$.*

If X is a space with the metric topology, then X is called metrizable space. Hence every metric space determines a metrizable space. However, it is possible to find several metrics d on X such that $\mathcal{T}_d = \mathcal{T}$ as shown in the following example.

Example 1.6 *Given a metrizable space $(X, \mathcal{T})$. Then it is always possible to find several metrics d on X such that $\mathcal{T}_d = \mathcal{T}$. For example, if we define $d'(x, y) = 2d(x, y)$. Then d' generates the same topology on X.*

Finally, we close theis section by looking at the union and intersection of two topologies.

Definition 1.7 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies defined on X. If each $\mathcal{T}_1-$open subset of X is also $\mathcal{T}_2-$open, i.e. $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_1$ is said to be coarser (smaller, weaker) than $\mathcal{T}_2$ or $\mathcal{T}_2$ is said to be finer (larger, stronger) than $\mathcal{T}_1$.*

Remark 1.8 *The intersection of two topologies is a topology, but thier union need not be a topology. To see this, for,* $X, \emptyset \in \mathcal{T}_1$ *and* $\mathcal{T}_2$ *and therefore belong to thier intersection. Suppose* A, B *are open sets in* $\mathcal{T}_1 \cap \mathcal{T}_2$*. Then, in particular,* $A, B \in \mathcal{T}_1$ *and* $A, B \in \mathcal{T}_2$*. Furtheremore, since* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are topologies,* AUB *and* $A \cap B$ *belong to both* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *and therefore to their intersection, which proves that* $\mathcal{T}_1 \cap \mathcal{T}_2$ *is a topology. This result can be generalized to any number of topologies and thus* $\cap_i \mathcal{T}_i$ *is also a topology on* X*. On the other hand, if* $X = \{a, b, c,\}$*,* $\mathcal{T}_1 = \{X, \emptyset, \{a\}\}$*, and* $\mathcal{T}_2 = \{X, \emptyset, \{b\}\}$*, then* $\mathcal{T}_1 \cup \mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b\}\}$ *which is not a topology since* $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$*.*

2 Neighbourhood and Neighbourhood systems

There are several ways of describing topologies but most of them are not convenient. in this section we shall present two most popular ways to describe topologies.

Definition 2.1 *(Neighbourhood at a point) A subset* A *of a topological space* X *is called a neighbourhood (hereafter abbreviated nbd) of a point* $x \in X$ *if there exists an open set* G *such that* $x \in G \subseteq A$

Remark 2.2 *Observe that* A *is a neighbourhood of a point* $x \in X$ *if and only if* $x \in \mathring{A}$*.*

If A is a neighbourhood of $x \in X$, the we may denote A by $\mathcal{N}_x$. Thus, it is very obvious that there may be several neighbourhoods of the point $x \in X$ and so we have the following definition:

Definition 2.3 *The class of all neighbourhoods of a point* $x \in X$ *is called the neighbourhood system of* x *and is denoted by* $\mathcal{N}(x)$*.*

Remark 2.4 *Observe that each open set containing the point* $x \in X$ *is a neighbourhood of* x*, usually called the open neighbourhood of* x*.*

Example 2.5 *Let* $x \in \mathbb{R}$*. Then the interval* $[x - \delta, x + \delta]$ *is a neighbourhood of* x*, since it contains the open interval* $(x - \delta, x + \delta)$ *which contains* x*.*

Example 2.6 *In an indiscrete space* X*,* X *is the only neighbourhood of each of its points. Therefore, for each* $x \in X$*,* $\mathcal{N}(x) = \{X\}$*.*

Example 2.7 *Let* $X = \{a, b, c\}$ *and* $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. *Then the neighbourhood systems of* a, b, c *are*

$\mathcal{N}(a) = \{\{a\}, \{a, b\}, \{a, c\}, X\}$.
$\mathcal{N}(b) = \{\{a, b\}, X\}$.
$\mathcal{N}(c) = \{X\}$.

The following result characterizes open sets in terms of neighbourhoods:

Theorem 2.8 *A subset* A *of a topological space* X *is open if and only if* A *is a neighbourhood of each of its points.*

Proof. Necessity. Suppose A is open, then $x \in A \subseteq A$ implies A is a nbd of each $x \in A$.

Sufficiency. Suppose A is a nbd of each $x \in A$. Then there exists an open set G_x such that $x \in G_x \subseteq A$. Then

$$A = \bigcup \{x : x \in A\} \subseteq \bigcup_{x \in A} G_x \subseteq A \quad \text{or} \quad A = \bigcup_{x \in A} G_x$$

shows that A is open and the proof is complete. ■

The following properties characterize the neighbourhood systems and may also be used to define a topology on X.

Theorem 2.9 *The following properties characterize the neighbourhood systems* $\mathcal{N}(x)$ *of* x *in a space* X.

1. $\mathcal{N}(x) \neq \emptyset$. *If* $A \in \mathcal{N}(x)$, *then* $x \in A$.
2. *If* $A, B \in \mathcal{N}(x)$, *then* $A \cap B \in \mathcal{N}(x)$.
3. *If* $A \in \mathcal{N}(x)$, $A \subseteq B$, *then* $B \in \mathcal{N}(x)$.
4. *If* $A \in \mathcal{N}(x)$, *then there exists* $B \in \mathcal{N}(x)$ *such that* $A \in \mathcal{N}(y)$, *for each* $y \in B$.

Proof. (1) is obvious.

To prove (2), let $A, B \in \mathcal{N}(x)$. Then there exist open sets G_1, G_2 such that $x \in G_1 \subseteq A$, $x \in G_2 \subseteq B$.Hence $x \in G_1 \cap G_2 \subseteq A \cap B$. Since $G_1 \cap G_2$ is open, this proves that $A \cap B \in \mathcal{N}(x)$.

To establish (3), let $A \in \mathcal{N}(x)$. Since $A \in \mathcal{N}(x)$, then $x \in G \subseteq A$, for some G open in X. Since $A \subseteq B$, then $x \in G \subseteq A \subseteq B$ which implies $x \in G \subseteq B$. This implies that $B \in \mathcal{N}(x)$.

To proof (4), since $A \in \mathcal{N}(x)$, then $x \in B \subseteq A$, for B open in X. $x \in B \subseteq B$ gives $B \in \mathcal{N}(x).If$ $y \in B$, then $B \subseteq A$ implies $A \in \mathcal{N}(y)$, for each $y \in B$.

Conversely, define $\mathcal{T} = \{A \subseteq X : A \in \mathcal{N}(x), \text{ for all } x \in A\} \cup \{\emptyset\}$, then it is easily seen that $\mathcal{T}$ is the desired topology on X for which $\mathcal{N}(x)$ is the neighbourhood system at $x \in X$. ■

3 Subspaces induced Topology. Bases and subbases

3.1 Subspaces induced Topology

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. It is possible to assign several topologies to A without reference to $\mathcal{T}$. However, in this section, we are interested in assigning a definite topology to A which A inherits from its parent space $(X, \mathcal{T})$. We shall discuss such a topology, called relative (or induced) topology and some interesting properties of the resulting space, called a subspace.

Definition 3.1 ***(Relative Topology)*** *Let A be a subset of a space $(X, \mathcal{T}_X)$. We assign a topology $\mathcal{T}_A$ to A in the following natural ways:*

$$\mathcal{T}_A = \{A \cap G : G \in \mathcal{T}_X\}.$$

$\mathcal{T}_A$ is called the relative topology on A and $(A, \mathcal{T}_A)$ is called a subspace of of a topological space $X, \mathcal{T}_X)$.

Claim 3.2 *We claim that $\mathcal{T}_A$ is a topology on A. (verify this claim)*

Example 3.3 *Let $X = \{a, b, c, d, e\}$ with $\mathcal{T}_X = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$. Let $A = \{a, d, e\}$. Then*

$$\mathcal{T}_A = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{d, e\}, A\}.$$

Remark 3.4 *A set may be open in the relative topology and may not be open in the parent topology. To see this, consider the last example and note that $\{d\}$ is open*

in A but not open in X.

4 Continuity in Topological Spaces

In this section, we define continuous map in a topological space and prove its several properties and characterization..

Definition 4.1 *Let (X, τ) and (Y, ρ) be topological spaces an f is a map from X to Y. Then f is said to be continuous if $f^{-1}(G)$ is open in (X, τ) whenever G is open in (Y, ρ).*

We can give a modified form of this definition by considering neighbourhoods instead of open sets. Indeed, we have

Definition 4.2 *Let $f : X \to Y$ be a map from a topological space X into a topological space Y. Then f is said to be continuous at a point $x \in X$ if and only if for*

each neighbourhood V of $f(x)$ in Y, there exists a neighbourhood U of $x \in X$ such that $f(U) \sqsubseteq V$.

Remark 4.3 *Observe that neighbourhood can be replaced by open neighbourhood in the above definition.*

Next, we give some characterizations of continuous map in the following theorem:

Theorem 4.4 *Let $f : X \to Y$ be a map from a topological space X into a topological space Y. Then the following conditions are equivalent:*

(i) f is continuous

(ii) For each open set A in Y, $f^{-1}(A)$ is open in X.

(iii) For each closed set B in Y, $f^{-1}(B)$ is closed in X.

(iv) For each open set $A \sqsubseteq X$, $f^{-1}(\overline{A}) \sqsubseteq \overline{f(A)}$

(v) For each open set $B \sqsubseteq Y$, $f^{-1}(B^o) \sqsubseteq (f^{-1}(B))^o$

(vi) For each open set $C \sqsubseteq Y$, $\overline{f^{-1}(C)} \sqsubseteq f^{-1}(\overline{C})$.

Proof. $(i) \implies (ii)$.If A is open in Y, then for each $x \in f^{-1}(A)$, $f(x) \in A$. This implies that A is an open neighbourhood of $f(x)$.

By continuity of f, there exists an open neighbourhood B of $x \in X$ such that $f(B) \sqsubseteq A$ or $B \sqsubseteq f^{-1}(A)$. Thus $x \in B \sqsubseteq f^{-1}(A)$. This implies that

$f^{-1}(A)$ is open in X.

$(ii) \implies (iii)$. This follows from the fact that $f^{-1}(Y - B) = X - f^{-1}(B)$, for every $B \sqsubseteq Y$.

$(iii) \implies (iv)$. We know that $f(A) \sqsubseteq \overline{f(A)}$ and so $A \sqsubseteq f^{-1}f(A) \sqsubseteq f^{-1}(\overline{f(A)})$ or $A \sqsubseteq f^{-1}(\overline{f(A)})$. By $(iii), \sqsubseteq f^{-1}(\overline{f(A)})$ is closed in X containing A.Hence $\overline{A} \sqsubseteq f^{-1}(\overline{f(A)})$

or $f(\overline{A}) \sqsubseteq \overline{f(A)}$.

$(iv) \implies (v)$. Given that $f(\overline{A}) \sqsubseteq \overline{f(A)}$, then take $A = X - f^{-1}(B^o) = f^{-1}(Y - B^o), B \sqsubseteq Y$. Then $f(A) \sqsubseteq Y - B^o, f(\overline{A}) \sqsubseteq \overline{f(A)}$ gives

$$\overline{A} \sqsubseteq f^{-1}(\overline{f(A)}) \sqsubseteq f^{-1}(\overline{Y - B^o}) = f^{-1}(Y - B^o) = A$$

gives $A = f^{-1}(Y - B^o) = X - f^{-1}(B^o)$ is closed. This implies that $f^{-1}(B^o)$ is open in X. Clearly, $f^{-1}(B^o) \sqsubseteq f^{-1}(B)$ implying that $f^{-1}(B^o) \sqsubseteq (f^{-1}(B))^o$.

$(v) \implies (vi)$. Take $B = Y - \overline{C} \sqsubseteq Y - \overline{C}$. Then $B^o = Y - \overline{C}$. Hence $f^{-1}(B^o) \sqsubseteq (f^{-1}(B))^o$ gives $f^{-1}(Y - \overline{C}) \sqsubseteq (f^{-1}(Y - \overline{C}))^o \sqsubseteq (f^{-1}(Y - C))^o$

$= (X - f^{-1}(C))^o = X - \overline{f^{-1}(C)}$ or $f^{-1}(Y - \overline{C}) \sqsubseteq X - \overline{f^{-1}(C)}$ or $X - f^{-1}(\overline{C}) \sqsubseteq X - \overline{f^{-1}(C)}$ or $\overline{f^{-1}(C)} \sqsubseteq f^{-1}(\overline{C})$.

$(vi) \implies (i)$. Let $x \in X$ and V an open neighbourhood of $f(x)$ in Y. Put $C = Y - V$. Then C is closed implies that $C = \overline{C}$ and $f(x) \notin C$. Then $\overline{f^{-1}(C)} \sqsubseteq f^{-1}(\overline{C})$

gives $\overline{f^{-1}(C)} \sqsubseteq f^{-1}(C) \implies f^{-1}(C)$ is closed. Next, put $U = X - f^{-1}(C)$ and therefore $x \in U$. Then U is an open neighbourhood of x and $U = X - f^{-1}(C)$ gives

$U = f^{-1}(Y - C) = f^{-1}(V)$ or $f(U) = f\ f^{-1}(V) \sqsubseteq V$ or $f^{-1}(U) \sqsubseteq V$. This proves that f is continuous. Hence the proof is complete. ■

As an application of (ii) in the above characterization we have:

Theorem 4.5 *If $f : f : (X, \tau) \to (Y, \rho)$ and $g : (Y, \rho) \to (Z, \sigma)$ are continuous, then $gof : (X, \tau) \to (Z, \sigma)$ is continuous.*

Proof. *Easy and left as an exercise.* ■

Theorem 4.6 *A map $f : (X, \tau) \to (Y, \rho)$ is continuous if and only if the inverse image of every member of a base for the topology on Y is open in X.*

Theorem 4.7 *A map $f : (X, \tau) \rightarrow (Y, \rho)$ is continuous if and only if the inverse image of every member of a subbase S for the topology on Y is open in X.*

Proof. Necessity. Suppose f is continuous. Then the inverse image of all open sets including the members of the subbase S are open in X.

Sufficiency. Suppose $f^{-1}(A) \in \tau$, for every $A \in S$. We shall show that f is continuous. Let $U \in \rho$. Then by definition of the subbase

$$U = \cup_i (A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_{n_i}}), \quad \text{where } A_{i_k} \in S.$$

Hence

$$\begin{aligned} f^{-1}(U) &= f^{-1} \cup_i (A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_{n_i}}) \\ &= \cup_i f^{-1}(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_{n_i}}) \\ &= \cup_i (f^{-1}(A_{i_1}) \cap f^{-1}(A_{i_2}) \cap ... \cap f^{-1}(A_{i_{n_i}})) \ \in \tau, \end{aligned}$$

since for each $A_{i_k} \in S$, $f^{-1}(A_{i_k}) \in \tau$. Accordingly, f is continuous and the proof is complete. ■

Definition 4.8 *A mapping $f : (X, \tau) \rightarrow (Y, \rho)$, taking a topological space (X, τ) into a topological space (Y, ρ) is called an open (respectively closed) mapping*

if $f(G)$ is open (respectively closed) in (Y, ρ) whenever G is open (respectively closed) in (X, τ).

Example 4.9 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ in the usual topology be defined by $f(x) = 1$, $\forall x \in \mathbb{R}$. Then, f is clearly a closed map (why)?. Observe that f is not open (why)?. Note that f*

is continuous.

Example 4.10 *The identity map $i : (X, \tau) \rightarrow (Y, \rho)$ is continuous (respectively open) if and only if $\rho \sqsubseteq \tau$ (respectively $\tau \sqsubseteq \rho$). If i is continuous (respectively open), then it*

is not necessarily open (respectively continuous).

Example 4.11 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ in the usual topology be defined by $f(x) = x^2$, $\forall x \in \mathbb{R}$. Then, f is not open but continuous as well as closed. To see this,*

consider $A = (-1, 1)$, then $f(A) = [0, 1)$ this implies that f is not open but f is continuous and closed. (verify).

Remark 4.12 *The above examples clearly show that continuous map, open map and closed map are independent notions.*

Definition 4.13 *(**Homeomorphism**) A map $f : (X, \tau) \to (Y, \rho)$, taking a topological space (X, τ) into a topological space (Y, ρ) is called a homeomorphism if and only if*

the following two conditions are satisfied:

(i) *f is bijective (that is, one-to-one and onto)*

(ii) *f and f^{-1} are continuous (that is f is bicontinuous).*

Alternatively, a homeomorphism is a one-one, open, continuous mapping of one topological space (X, τ) onto another topological space (Y, ρ).

Two spaces are topologically equivalent (homeomorphic), written $X \simeq Y$ if there exists a homeomorphism of one space onto the other.

Thus if (X, τ) and (Y, ρ) homeomorphic, their points can be put into one-to-one correspondence in such a way that their open sets also correspond to each other.

Example 4.14 *(1). Let $i : (X, \tau) \to (X, \tau)$ be the identity map. then i is a homeomorphism.*

(2). Let $f : X \to Y$ be a bijective map from the discrete space X into a discrete space Y. Then f is a homeomorphism.

(3). Let $X = \{1, 2\}$, $Y = \{a, b\}$, $\mathcal{T}_X = \{\emptyset, \{1\}, X\}$, $\mathcal{T}_Y = \{\emptyset, \{a\}, Y\}$. Let $f : X \to Y$ be defined by $f(1) = a$, $f(2) = b$. Then f is a homeomorphism.

The following result gives the several characterizations of homeomorphism:

Theorem 4.15 *Let $f : X \to Y$ be bijective. Then the following are equivalent:*

Definition 4.16

(i) *f and f^{-1} are continuous*

(ii) *f is continuous and open*

(iii) *f is continuous and closed*

(iv) $f(\overline{A}) = \overline{f(A)}$, $A \sqsubseteq X$.

Proof. Left as exercise. ■

Definition 4.17 *(Topological Property) A topological property is a property which, if possessed by a topological space, is also possessed by all topological spaces homeomorphic to that space.*

We shall end this section and in particular, topological property with the following examples:

Example 4.18 *Let $X = (-1, 1)$ and $f : X \to \mathbb{R}$ be defined by $f(x) = \tan \frac{\pi x}{2}$. Then f is a homeomorphism and hence $(-1, 1) \simeq \mathbb{R}$. Observe that $(-1, 1), \mathbb{R}$ have different lenghts, therefore "lengh" is not a topological property. Note also that X is bounded and $\mathbb{R}$ is not bounded, therefore,'boundedness' is not a topological property. '*

Example 4.19 *Straightness is not a topological property, for a line may be bent and stretched until it is wiggly. Being "triangular' is not a topological property since a triangle can be continuously deformed into a cicle and conversely. However, limit point, interior point, boundary, neighbourhood and first/second countability compactness and connectedness are topological properties.*

Some of these topological properties will be studied in the subsequent sections.

5 Separable Spaces and Some Separation Axioms

In this section, we will study some separation axioms that will enable us state exactly that a given topology has a reasonable number of open sets to serve our purpose.

Definition 5.1 *A subset A of a topological space X is said to be everywhere dense or dense if and only if $\bar{A} = X$.*

Example 5.2 *The set of rational numbers* $\mathbb{Q}$ *is dense in* $\mathbb{R}$. *Observe that in the case of the indiscrete space, every nonempty subset is dense in* X.

Definition 5.3 *A subset* A *of a topological space* X *is said to be nowhere dense if the interior of the closure of* A *is empty. That is,* $\left(\bar{A}\right)^0 = \emptyset$.

Example 5.4 *Take* $A = \{0 < x < 1,\ x \in \mathbb{Q}\}$, *then* $\bar{A} = [0,1]$ *and so* $\left(\bar{A}\right)^0 = (0,1) \neq \emptyset$. *Therefore,* A *is not nowhere dense in* $\mathbb{R}$. *If we take* $A = \left\{\frac{1}{n}\right\}$, *then* $\bar{A} = \left\{0, 1, \frac{1}{2}, ...]\right\}$ *and so* $\left(\bar{A}\right)^0 = \emptyset$. *Hence,* A *is nowhere dense in* $\mathbb{R}$.

Definition 5.5 *A topological space* X *is said to be separable if it contains a countable dense subsets.*

Example 5.6 $\mathbb{R}$ *with the usual topology is separable (since the set* $\mathbb{Q}$ *is countable dense subset of* $\mathbb{R}$*). However,*$\mathbb{R}$ *with the discrete topology is not separable since every subset of* $\mathbb{R}$ *is both opoen and closed relative to the topology and so the only dense subset of* $\mathbb{R}$ *is* $\mathbb{R}$ *itself and it is not countable.*

5.1 $T_i - Spaces$

Definition 5.7 (T_0-*Spaces*) *A topological space* X *is called a* T_0-*space if and only if for every pair of distinct points* $x, y \in X$, *there exists an open set* G *containing one of them but not the other.*

Example 5.8 *(i) Every subspace of a* T_0-*space is also a* T_0-*space. To see this, if* Y *is a subspace of a* T_0-*space* X, *then foe each pair of distinct points* $x, y \in Y$, *there exists an open set* G *containing* x *but not* y, *since* $x, y \in X$ *also.*

(ii) Let $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ *.Then*

$$\begin{aligned} a, b &: \quad a \in \{a\}, b \notin \{a\} \\ b, c &: \quad b \in \{b\}, c \notin \{b\} \\ a, b &: \quad a \in \{a\}, c \notin \{a\}. \end{aligned}$$

This shows that X *is a* T_0-*space.*

5.2 T_1-Space

Definition 5.9 *(T_1-Spaces) A topological space X is called a T_1-space if and only if for every pair of distinct points $x, y \in X$, ech belongs to an open set which does*

not contain the other. That is, $x, y \in X$ implies that there exist open sets G, H such that $x \in G, x \notin H$ and $y \in H$, $y \notin G$.

Remark 5.10 *Every subspace of a T_1-space is a T_1-space. To see this, let Y be a subspace of a T_1-space X, then $x, y \in Y \Longrightarrow x, y \in X$, hence $\exists$ open sets G, H such that $x \in G, x \in H$, and $y \in H, y \notin G$. This implies that $x \in G \cap Y$, $x \notin H \cap Y$ and $y \in H \cap Y$, $y \notin G \cap Y$. This implies that (Y, τ_Y) is a T_1-space.*

Example 5.11 *Let $X = \{a, b, c\}$ and define*

$$\begin{aligned} \tau_1 &= \{X, \emptyset, \{a\}\} \\ \tau_2 &= \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}. \end{aligned}$$

Then (X, τ_1) is not a T_1-space, since b and c belong only to X which also contains a, whereas (X, τ_2) is not a T_1-space.

Remark 5.12 *Every T_1-space is a T_0-space but not conversely. To see this, take $X = \{1, 2\}$ and $\tau = \{\emptyset, \{1\}, X\}$, then X is a T_0-space but not a T_1-space.*

The following result gives a characterization of a T_1-space:

Theorem 5.13 *A topological space X is a T_1-space if and only if every singleton set in X is closed.*

Proof. Necessity Let X be a T_1-space . We shall show that each singleton $\{x\}$ is closed by showing that $X - \{x\}$ is open.

Now, let $y \in X - \{x\}$, $x \neq y$. Then $\exists$ an open set H such that $y \in H$ and $x \notin H$. Then $y \in H \subseteq X - \{x\}$ and hence $\cup \{H : y \in X - \{x\}\} = X - \{x\}$.

This implies that $X - \{x\}$ is open.

Sufficiency. Suppose that each $\{x\}$ is closed. We shall show that X is a T_1-space. Let $x, y \in X$, $x \neq y$. Then $X - \{x\}$, $X - \{y\}$ are open in X.

Hence, $x \in X - \{y\}$, $y \notin X - \{y\}$ and $x \notin X - \{x\}$, $y \in X - \{x\}$. This implies that X is a T_1-space and the proof is complete. ■

5.3 T_2-Space or Hausdorff Space

Definition 5.14 *(T_2-Spaces) A topological space X is called a T_2-space or Hausdorff space if and only if for every pair of distinct points $x, y \in X$, there exist open*

sets G, H such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$.

Remark 5.15 *Every subspace of a T_{2-}space is a T_{2-}space . To see this, let Y be a subspace of a T_{2-}space X. TLet $x, y \in Y$. Then $x, y \in X$ also and since X is*

a T_{2-}space, there exist open sets G, H such that $x \in G$, $y \in H$ and $G \cap H = \emptyset$. But $G \cap Y$, $H \cap Y$ are open in τ_Y

and $(G \cap Y) \cap (H \cap Y) = Y \cap (G \cap H) = Y \cap \emptyset = \emptyset$.

Example 5.16 *Every metric space is a Hausdorff space. To see this, let (X, d) be a metric space and let $x, y \in (X, d)$. Now consider the open sphere G, H with*

centres x, y and radius $\frac{\epsilon}{3}$. Since $x \neq y$,then $d(x, y) = \epsilon > 0$. Suppose $G \cap H \neq \emptyset$. Let $z \in G \cap H$, then $d(x, z) < \frac{\epsilon}{3}$, $d(y, z) < \frac{\epsilon}{3}$ and so by triangle

inequality we have

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}$$

which contradicts the fact that $d(x, y) = \epsilon$. Hence $G \cap H = \emptyset$ and so (X, d) is Hausdorff.

Remark 5.17 *Every T_2-space is a T_1-space but not conversely as the following example shows: Let $\mathbb{R}$ be with the cofinite topology. Then $\mathbb{R}$ is a T_1-space but*

not a T_2-space.

Example 5.18 *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, X\}$. Then X is a T_2-space as well as a T_1-space.*

Remark 5.19 *Observe from Remarks 5.12 and 5.17 that $T_2 \Longrightarrow T_1 \Longrightarrow T_0$ but the reverse implications do not hold.*

5.4 Regular and Normal Spaces

Definition 5.20 *A topological space X is said to be regular if for every closed subset F of X and for every $x \in X$, $x \notin F$, there exists disjoint open sets G, H such*

that $x \in G$ and $F \subset H$.

Remark 5.21 *Every subspace of a regular space is regular. To see this, let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$. Clearly, the class ϱ of closed sets is given*

by $\varrho = \{\emptyset, X, \{b, c\}, \{a\}\}$. Then (X, τ) is a regular space but it is not a T_1-space since $\{b\}$ and $\{c\}$ are not closed sets.

Definition 5.22 *A regular space which is also a T_1-space is called a T_3-space.*

Theorem 5.23 *Every T_3-space is a T_2-space.*

Proof. Let X be a T_3-space. Since X is a T_1-space, if $x \in X$, $\{x\}$ is a closed set. If $x \notin y$, then by the regularity of X, there exist open sets G, H such

that $y \in G$, $\{x\} \subset H$ and $G \cap H = \emptyset$. This implies that every pair of distinct elements $x, y \in X$ satisfies the criterion for a space to be Hausdorff.

Hence, X is a T_2-space and the proof is complete. ■

Definition 5.24 *A topological space X is said to be completely regular if and only if for every closed subset F of X and for every $x \in X$, $x \notin F$, there exists a*

continuous real valued function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.

Claim 5.25 *Every subspace of a completely regular space is completely regular.*

Theorem 5.26 *Every completely regular space is regular.*

Proof. Assume that X is completely regular. Then, let F be a closed subset of X and let $x \in X$, $x \notin F$. Then, there exists a real valued function $f : X \to [0, 1]$

such that $f(x) = 0$ and $f(F) = 1$. Since the real line $\mathbb{R}$ is Hausdorff, the subspace $[0, 1]$ of $\mathbb{R}$ is also Hausdorff. Hence, there exist open sets G, H such that $0 \in G$, $1 \in H$, $G \cap H = \emptyset$. But $f^{-1}(G)$ and $f^{-1}(H)$ are open in X since f is continuous and their intersection is empty. Hence, X is a regular space. ■

Definition 5.27 *A completely regular space which is also a T_1-space is called a Tychonoff space.*

Definition 5.28 *Let $F = \{f_i\}$ be a class of functions from any set X to a set Y. Then F is said to separate points if and only if every pair of distinct points $x, y \in X$,*

there exists an $f_i \in F$ such that $f_i(x) \neq f_i(y)$.

Remark 5.29 *Observe that if $F = \{\sin x, \ \sin 2x, \ \sin 3x, \ ...\}$ is a class of functions on $\mathbb{R}$, then $f_i(0) = f_i(\pi) = 0$. So F does not separate points of $\mathbb{R}$.*

Theorem 5.30 *The space $\varrho(X, \mathbb{R})$ of all continuous real valued functions defined on a Tychonoff space X separates points.*

Proof. Let X be a Tychonoff space and let $x, y \in X$. Since X is a T_1-space, $\{x\}$ and $\{y\}$ are closed. Since$X$ is completely regular, there exists a real

valued continous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(\{y\}) = 1$ which implies that $f(x) \neq f(y)$. Hence $\varrho(X, \mathbb{R})$ separates points of X. ■

Definition 5.31 *A topological space X is said to be normal if and only if for every pair F_1, F_2 of disjoint closed subsets of X, there exists disjoint open sets G, H such*

that $F_1 \subset G$ and $F_2 \subset H$.

Example 5.32 *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, then the class ϱ of closed sets is given by $\varrho = \{\emptyset, X, \{b, c\}, \{a, c\}, \{c\}\}$. If F_1 and F_2 are disjoint*

closed subsets of X, one of them (say) F_1, must necessarily be $\emptyset$. Hence, $\emptyset$ and X are disjoint and open sets and so $F_1 \subset \emptyset$ and $F_2 \subset X$. Thus

(X, τ) is a normal space. But it is not a T_1-space since $\{a\}$ and $\{b\}$ are not closed; neither is it regular since $a \notin \{c\}$ and the only open set

containing c is X itself, which contains a.

Example 5.33 *(i) Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. *Then* X *is normal. Observe that* X *is neither* T_1 *nor regular*

(ii) Any space with the discrete topology or trivial topology is normal, for in the first case every subset is open and closed while in the

second case, the only two subsets are X *and* $\emptyset$, *which are both open and closed.*

Finally, we close this section by defining a T_4-space.

Definition 5.34 *A normal space which is also a* T_1-*space is called a* T_4-*space.*

Theorem 5.35 *Every* T_4-*space is a* T_3-*space.*

Proof. Let X be a T_4-space. Let $x \in X$ and let F be a closed subset of X disjoint from x. Since X is T_1-space, the singleton $\{x\}$ is closed. By the

normality condition on X, there exist open sets G, H such that $\{x\} \subset G$, $F \subset H$ and $G \cap H = \emptyset$. This implies that X is a T_3-space. ■

Lemma 5.36 *(Urysohn) Let* X *be a normal space. If* F_1 *and* F_2 *are closed subspaces of* X, *then there exists a real-valued continuous function*

$f : X \to [0, 1]$ *such that* $f(F_1) = 0$ *and* $f(F_2) = 1$.

Remark 5.37 *By virtue of Urysohn"s lemma, a* T_4-*space is a Tychonoff. Also, since a complete regular space is regular, a Tychonoff space is a* T_3-*space.*

6 Compactness

Definition 6.1 *Let* X *be a topological space. A class* $\{G_i\}$ *of open subsets of* X *is an open cover of* X *if each point in* X *belongs to at least on* G_i. *In other words,* $X \sqsubseteq \cup_i G_i$. *A subclass of an open cover, which is itself an open cover is called a subcover.*

Definition 6.2 *A topological space* X *is said to be compact if every open cover has a finite subcover. A compact subspace of a topological space is a subspace which is compact as a topological space in its own right.*

Example 6.3 *(i) The class* $\mathcal{G} = \{(-n, n) : \ n \in \mathbb{N}\}$ *and* $\mathcal{H} = \{(-2n, 2n) : \ n \in \mathbb{N}\}$ *are open covers of the real line* $\mathbb{R}$ *in the usual topology. Observe that* $\mathcal{H}$ *is a subcover of* $\mathcal{G}$.

(ii) The class $\mathcal{F} = \left\{(\frac{1}{n}, 1) : n = 2, 3, ...\right\}$ *is an open cover of* $(0, 1)$ *as a subspace of* $\mathbb{R}$.

Example 6.4 *Consider the class* $\{\mathcal{B}_p = p \in \mathbb{Z} \times \mathbb{Z}\}$, *where* $\mathcal{B}_p$ *is the open disc in* $\mathbb{R}^2$ *with centre* $p = (m, n)$ *and radius* $r = 1, \mathbb{Z}$ *is the set of integers. This class constitutes an open cover of* $\mathbb{R}^2$. *However, the class of open discs with centre* p *and radius* $\frac{1}{3}$ *will not cover* $\mathbb{R}^2$, *since there exist points like* $(\frac{1}{3}, \frac{1}{3})$ *which do not belong to any member of the class.*

Example 6.5

1. *All finite spaces are compact and may be referred to as trivial compact spaces.*
2. *Every cofinite space is compact*
3. $\mathbb{R}$ *with the usual topology is not compact*
4. *No infinite discrete space is compact*

Verify all the assertions in Example 6.5.

Theorem 6.6 *Any closed subspace of a compact space is compact.*

Proof. Let X be compact and let Y be a closed subspace of X. We shall show that Y is compact.

Let $\{G_i\}$ be an open cover of Y. Each G_i, being open in the relative topology on Y, is obtained as the intersection of Y with an open subsets U_i of X. Since Y is closed, Y^c is open and so Y^c together with the U_i'*s* constitutes an open cover of X. But X is compact and therefore this open cover admits a finite subcover. If Y^c occur in this subcover, we omit it and the remaining sets constitutes a finite subclass $\{U_{i_1}, U_{i_2}, ..., U_{i_m}\}$ whose union covers X. By taking the intersections of each of these U_{i_m}'*s* with Y, we obtain a finite subcover $\{G_{i_1}, G_{i_2}, ..., G_{i_m}\}$ of the original cover $\{G_i\}$ of Y which implies that Y is compact. ■

Theorem 6.7 *Any continuous image of a compact space is compact.*

Proof. Let $f : X \to Y$ be a continuous mapping taking a compact space X into an arbitrary topological space Y. Let the image of X under f be $f(X)$. We shall show that $f(X)$ is a compact subspace of Y. Let $\{G_i : i \in I\}$ be an open cover of $f(X)$.Then each $G_i = f(X) \cap H_i$, H_i is open in Y. Since f is continuous, $f^{-1}(H_i)$ is open in X, for each H_i and $\{f^{-1}(H_i) : i \in I\}$ forms an open cover of X. By the compactness of X, $\{f^{-1}(H_i) : i \in I\}$ has a finite subcover. The union of the corresponding H_i's, of which these are inverse images clearly contains $f(X)$ and therefore the associated G_i's constitute a finite subcover of the original open cover of $f(X)$. This implies that $f(X)$ is compact as a subspace of Y. ■

Definition 6.8 *A class of subsets $\mathcal{G} = \{G_i : i \in I\}$ of a nonempty set X is said to have the finite intersection property, if every finite subclass $\{G_{i_1}, G_{i_2}, ..., G_{i_m}\}$ of $\mathcal{G}$ has a nonempty intersection, that is, $G_{i_1} \cap G_{i_2} \cap ... \cap G_{i_m} \neq \emptyset$.*

Example 6.9 *Let $X = \mathbb{R}$ and $\mathcal{G} = \{..., (-\infty, -2), (-\infty, -1), (-\infty, 0), (-\infty, 1), (-\infty, 2), ...\}$ is a class of open intervals. Then $\mathcal{G}$ has a finite intersection property. Also, if $X = \mathbb{R}$ and $\mathcal{G} = \left\{(0, 1), (0, \frac{1}{2}), (0, \frac{1}{4}), ...\right\}$. Then $\mathcal{G}$ has a finite intersection property.*

Next, we characterize compactness in terms of closed sets as follows:

Theorem 6.10 ***(Characterization of Compactness)*** *The following statements are equivalent in a space*

1. *X is compact*
2. *Every class of closed sets with empty intersection has a finite subclass with empty intersection.*

Proof. (1) $\Longrightarrow$ (2). Suppose $\{F_i\}$ is a class of closed set with $\cap F_i = \emptyset$. Then by De-Morgan's law $X = X - \emptyset = X - \cap F_i = \cup(X - F_i)$. This implies that $\{X - F_i\}$ is an open cover of X. Since X is compact, therefore X has a finite cover $\{X - F_{i_1}, X - F_{i_2}, ..., X - F_{i_n}\}$, that is, $\cup_{j=1}^{n}(X - F_{i_j}) = X$. Again by De-Morgan's law $\cap_{j=1}^{n} F_{i_j} = X - X = \emptyset$. This gives (2).

(2) $\Longrightarrow$ (1). This is similar and hence left for the reader to verify. ■

Definition 6.11 *A subset A of a topological space X is said to be sequentially compact if and only if every sequence in A has a subsequence which converges to a point in A.*

Remark 6.12 *If A is a finite subset of X, then it is sequentially compact. For, if $\{x_n\}$ is a sequence in A, then at least one of the elements $x \in A$ must appear an infinite number of times in the sequence. Thus $\{x, x, ...\}$ is a subsequence of $\{x_n\}$ and it converges to $x \in A$. Also, the open interval $(0, 1)$ is not sequentially compact, since the sequence $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ converges to 0 and so does every subsequence. But $0 \notin (0, 1)$.*

Definition 6.13 *A subset A of a topological space X is countably compact if and only if every infinite subset B of A contains a limit point belonging to A.*

Example 6.14 *Every closed and bounded interval is countably compact.*

Remark 6.15 *Note that if B is an infinite subset of $A = [a, b]$, then B is also bounded and hence contains a limit point (by the Bolzano-Weierstrass theorem which states that every infinite bounded set bas at least one limit point) which belongs to A since A is closed. Observe that the open interval $(0, 1)$ is not countably compact since $B = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ has only one limit poin, 0, which does not belong to $(0, 1)$.*

Theorem 6.16 *Let X be a topological space. If X is compact or sequentially compact, then it is also countably compact.*

Proof. We shall show that compact $\Longrightarrow$ countably compact $\Longleftarrow$ sequentially compact. First, assume that X is compact. Let A be a subset of X with no limit point belonging to X. Then each point $x \in X$ belongs to an open set G_X which contains at most one point of A.The class $\{G_X\}$ is an open cover of X and by the compactness of X, there exists a finite subcover $\{G_{X_1}, G_{X_2}, ..., G_{X_m}\}$ such that $A \subset X \subseteq \cup_{i=1}^{m} G_{X_i}$. But each G_{X_i}.contains at most one point of A and hence being a subset of $\cup_{i=1}^{m} G_{X_i}$ can contain at most m points which in turn implies that A is finite. Thus, every infinite subset of X will contain at least one limit point in X which proves that X is countably compact. Next, suppose X is sequentially compact and let A be an infinite subset of X. Then there exists a sequence $\{x_n\} \in A$ with distinct terms and this contains a subsequence $\{x_{n_i}\}$, also with distinct terms. This subsequence converges to a point $x \in X$. Hence, every open set containing x, contains an infinite number of points of A. Accordinly, $x \in X$ is a limit point of A which implies that X is countably compact. ■

Remark 6.17 *The continuous image of a sequentially (countably) compact set is sequentially (countably) compact.*

We end this section by looking briefly at local compactness.

Definition 6.18 *A topological space X is said to be locally compact (briefly $\mathcal{L}$-compact) at a point $x \in X$ if and only if x has a compact neighbourhood in X.If X is $\mathcal{L}$-compact at every point, then X is called a locally compact space.*

Example 6.19 *Compact spaces are $\mathcal{L}$-compact. Suppose X is compact. X is a neighbourhood of each of its points implies X is $\mathcal{L}$-compact.Thus, we have the following result.*

Theorem 6.20 *A compact space is $\mathcal{L}$-compact.*

Example 6.21 $\mathbb{R}$ *with the usual topology is $\mathcal{L}$-compact, since for each $x\mathbb{R}$, we have $(a, b) \subseteq [a, b]$. Thus $[a, b]$ is a neighbourhood of x which is compact by Heine-Borel theorem. This proves that $\mathbb{R}$ is $\mathcal{L}$-compact. But recall that $\mathbb{R}$ is not compact.*

Remark 6.22 *The above example clearly shows that a locally compact space may not be compact. Therefore, the class of compact spaces is a subclass of the class of $\mathcal{L}$-compact spaces. Hence the notion of local compactness generalizes the notion of compactness.*

7 Connectedness

Definition 7.1 *(Connected Space) Let X be a topological space. Then X is said to be disconnected if and only if there exist nonempty disjoint open sets G and H such that $X = G \cup H$ and $G \cap H = \emptyset$. X is said to be connected if and only if it cannot be expressed as the union of two disjoint, nonempty open (closed) sets.*

Example 7.2

1. *$(0, 1) - \left\{\frac{1}{3}\right\}$ is disconnected*
2. *$\{x\} \subseteq \mathbb{R}$ is connected*
3. *Every indiscrete space is connected*

4. *Every discrete space with more than one point is disconnected*

5. *$\mathbb{R}$ with the upper limit topology is disconnected, since $\{x : x > a\}$ and $\{x : x \leq a\}$ are both open sets which forms a disconnection of $\mathbb{R}$.*

Example 7.3 *Let $X = \{0, 1\}$ and $\tau = \{\emptyset, \{0\}, X\}$. Then X is connected. This space is called the Sierpinski space.*

Example 7.4 *Let $X = \{a, b, c\}$ and $\tau_1 = \{\emptyset, \{c\}, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ Then (X, τ_1) is disconnected while (X, τ_2) is connected.*

Definition 7.5 *A subset A of a topological space X is said to be disconnected if there exist open sets G, H of X such that $A \cap G$ and $A \cap H$ are disjoint nonempty sets whose union is A. We say that $G \cup H$ is a disconnection of A.*

Definition 7.6 *A subset A of a topological space X which is not connected is said to be disconnected.*

The characterizations of connected spaces are given in the following result:

Theorem 7.7 *Let X be a topological space, then the followings are equivalent:*

1. *X is connected*

2. *The only open and closed subsets of X are $\emptyset$, X*

3. *There does not exist a continuous map $f : X \to \{0, 1\}$ from a space X onto the discrete space $\{0, 1\}$.*

Proof. We this theorem by contrapositive method.

$\sim (2) \implies \sim (1)$. Suppose $A \subseteq X$ that is both open and closed, and $A \neq \emptyset$, $A \neq X$. Then $X = A \cup (X - A)$ gives a disconnection of X.

$\sim (3) \implies \sim (2)$. Suppose there is a continuous map $f : X \to \{0, 1\}$ from a space X onto the discrete space $\{0, 1\}$. Then $f^{-1}\{0\}$ and $f^{-1}\{1\}$ are open and $X = f^{-1}\{0\} \cup f^{-1}\{1\}$. Thus $f^{-1}\{0\}$, $f^{-1}\{1\}$ are the nonempty open and closed subsets of X.

$\sim (1) \Longrightarrow \sim (3)$. If $X = A \cup B$, A, B are nonempty disjoint open sets. Then define $g : X \to \{0, 1\}$ as

$$g(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then g is continuous surjective map. This completes the proof. ■

The above theorem can be stated in the following form:

Theorem 7.8 *Let X be a topological space, then the followings are equivalent:*

1' *X is disconnected*

2' *There exists a nonempty proper closed and open subset of X*

3' *There exists a continuous map $f : X \to \{0, 1\}$ from a space X onto the discrete space $\{0, 1\}$.*

Now we study the invariance properties of connected spaces.

Theorem 7.9 *Any continuous image of a connected space is connected.*

Proof. Let $f : X \to Y$ be a continuous mapping of a connected space X into an arbitrary topological space Y. We have to show that $f(X)$ is a connected subspace of Y. Suppose $f(X)$ is disconnected. Then, there exist open sets G, H of Y such that $f(X) \subset G \cup H$, $G \cap H \subset (f(X))^c$,$f(X) \cap G \neq \emptyset$, $f(X) \cap H \neq \emptyset$. As f is continuous, $f^{-1}\{G\}$ and $f^{-1}\{H\}$ are open sets of X and $f^{-1}\{G\} \cup f^{-1}\{H\} = X$ gives a disconnection of X, which contradicts the connectedness of X. Thus $f(X)$ is connected as a subspace of Y. ■

Theorem 7.10 *A topological space X is said to be disconnected if and only if there exists a continuous mapping f of X onto the discrete two-point space $\{0, 1\}$.*

Proof. Suppose X is disconnected. Then, there exist open sets G, H such that $X = (X \cap G) \cup (X \cap H)$. Define the map f by

$$f(x) = \begin{cases} 1 & \text{if } x \in (X \cap H) \\ 0 & \text{if } x \notin (X \cap G). \end{cases}$$

Clearly, f is continuous and onto since $(X \cap G)$ and $(X \cap G)$ are nonempty, open and disjoint.

Conversely, if there exists such a continuous mapping of X onto $\{0,1\}$, then X is disconnected. For, if X were connected, f being continuous, its image $\{0,1\}$ should be connected by the last theorem and $\{0,1\}$ is certainly disconnected. ■