PHS 311

ANALYTICAL MECHANICS I

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Course Outline

1. Newtonian Mechanics
2. Conservative Forces and Potentials, Central Force Problems
3. Oscillations
4. Collision of Particles, Moving Frame of Reference and Elementary Mechanics of Rigid Bodies

CHAPTER ONE

1.0 Newtonian Mechanics – Motion of A Particle in One, Two and Three Dimensions

$$v = \frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

$$v = |v| = \left|\frac{dr}{dt}\right| = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)} = \frac{ds}{dt}$$

1.1 Linear Acceleration

$$a = \frac{dv}{dt}$$

$$a = \frac{d^2 r}{dt^2} = \frac{d^2 x}{dt^2}i + \frac{d^2 y}{dt^2}j + \frac{d^2 z}{dt^2}k$$

$$a = |a| = \sqrt{\left(\left(\frac{d^2 x}{dt^2}\right)^2 + \left(\frac{d^2 y}{dt^2}\right)^2 + \left(\frac{d^2 z}{dt^2}\right)^2\right)}$$

Newtonian Mechanics hinge on the three laws of Newton's:

1. Every particle continues in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. If F is the (external) force acting on a particle of mass m which as a consequence is moving with velocity v, then

   $F = \frac{d}{dt}(mv) = \frac{dP}{dt}$ where P = mv is the momentum.

   If m is independent of time t, we have $F = m\frac{dv}{dt}$, F = ma where a is the acceleration of the particle.

3. If particle A acts on particle B with a force F_{AB} in a direction along the line joining the particles, while particle B acts on particle A with a force F_{BA}, then $F_{BA} = -F_{AB}$. In other words, to every action there is an equal and opposite reaction.

Example

Due to a force field, a particle of mass 5 units moves along a space curve whose position vector is given as a function of time t by

$$r = \left(2t^3 + t\right)i + \left(3t^4 - t^2 + 8\right)j - 12t^2 k$$

Find (a) the velocity, (b) the momentum, (c) the acceleration and (d) the force field at any time t

Solution

(a) Velocity $v = \frac{dr}{dt}$

$$v = \left(6t^2 + 1\right)i + \left(12t^3 - 2t\right)j - 24tk$$

(b) Momentum P = mv

P = 5v

$$P = \left[\left(6t^2 + 1\right)i + \left(12t^3 - 2t\right)j - 24tk\right]$$

$$P = \left(30t^2 + 5\right)i + \left(60t^3 - 10t\right)j - 120tk$$

(c) Acceleration $a = \frac{dv}{dt} = \frac{d^2 r}{dt^2}$

$$a = 12ti + \left(36t^2 - 2\right)j - 24k$$

(d) Force $F = \frac{dP}{dt} = m\frac{dv}{dt}$

$$F = 60ti + \left(180t^2 - 10\right)j - 120k$$

1.2 Work

If a force F acting on a particle gives it a displacement dr, then the work done by the force on the particle is defined as $dW = F.dr$

The total work done W in moving the particle from point A to point B along a curve C is given by

$$W = \int_C F.dr = \int_A^B F.dr = \int_{r_1}^{r_2} F.dr$$

r_1 and r_2 are position vectors of A and B respectively.

1.3 Power

This is the time rate of doing work on a particle. Let P represent Power and W represent Work.

$$P = \frac{dW}{dt}$$

If F is the force acting on a particle and v is the velocity of the particle, then we have $P = F.v$

1.4 Kinetic Energy

If a particle has a constant mass and at times t_1 and t_2 it is located at A and B and it moves with velocities $v_1 = \frac{dr_1}{dt}$ and $v_2 = \frac{dr_2}{dt}$ respectively. Then the total work done in moving the particle along C from A to B is given by

$$W = \int_C F.dr = \int_{r_1}^{r_2} F.dr = m\int_{r_1}^{r_2} \frac{dv}{dt}.dr$$

$$W = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

Let $T = \frac{1}{2}mv^2$, the kinetic energy of the system. Then, the total work done from A to B along curve C equals

W = kinetic energy at B – kinetic energy at A

$W = T_2 - T_1$

$$T_1 = \frac{1}{2}mv_1^2,\ T_2 = \frac{1}{2}mv_2^2$$

Example

Find the work done in moving an object along a vector r = 3i + 2j – 5k if the applied force is F = 2i – j – k.

Solution

Work done W = F.dr

W = (2i – j – k).(3i + 2j – 5k)
W = 6 – 2 + 5
Work done W = 9J

CHAPTER TWO

2.0 Conservative Forces

A force is conservative if its dependence on the position vector r of the particle is such that the work W can always be expressed as the difference between the quantity $V_p(r)$ at the initial point and its value at the final point.

Let us suppose that there exists a scalar function V such that $F = -\nabla V$. Then the following can be proved.

Theorem 1: The total work done in moving the particle along C from P_1 to P_2 is

$$W = \int_{P_1}^{P_2} F.dr = V(P_1) - V(P_2)$$

The work done is independent of the path C joining points P_1 and P_2.

If the work done by a force field in moving a particle from one point to another point is independent of the path joining the points then the force field is said to conservative.

Theorem 2: A force field F is conservative if and only if there exists a continuously differentiable scalar field V such that $F = -\nabla V$ or equivalently, if and only if

$\nabla \wedge F = CurlF = 0$ identically.

Theorem 3: A continuously differentiable force field F is conservative if and only if for any closed non-intersecting curve C (simple closed curve)

$$\oint F.dr = 0$$

that is, the total work done in moving a particle around any closed path is zero.

2.1 Potential Energy or Potential

$$F = -\nabla V$$

V is called the potential energy, it is also called the scalar potential or potential. Then, total work done from P_1 to P_2 along C = Potential energy at P_1 – Potential energy at P_2

$$W = V_1 - V_2$$

$$W = V(P_1) - V(P_2)$$

V can be expressed as $V = \int_{r_o}^{r} F.dr$

2.2 Conservation of Energy

For a conservative field, $T_2 - T_1 = V_1 - V_2$

$T_1 + V_1 = T_2 + V_2$

But $T_1 = \frac{1}{2}mv_1^2$ and $T_2 = \frac{1}{2}mv_2^2$

So, $\frac{1}{2}mv_1^2 + V_1 = \frac{1}{2}mv_2^2 + V_2$ *

E = T + V = Total energy. This is the sum of the kinetic energy and potential energy. From equation *, the total energy at P_1 is the same as the total energy at P_2. Hence,

T + V = constant = E

Theorem: In a conservative force field the total energy (that is, sum of kinetic energy and potential energy) is a constant. This is the principle of conservation of energy.

2.3 Impulse

$$I = \int_{t_1}^{t_2} F.dt$$

This is also equal to change in momentum

$$I = \int_{t_1}^{t_2} F.dt = mv_2 - mv_1 = P_2 - P_1$$

2.4 Torque and Angular Momentum

The torque of a particle with position vector r which moves in a force field F is define as

$$T = r \wedge F$$

The torque of a particle is the moment of the force F about reference point say O. The magnitude of T is a measure of the 'turning effect' produced on the particle by the force. We obtain the angular momentum from the torque.

$$r \wedge F = \frac{d}{dt}\left[m(r \wedge v)\right] = m(r \wedge v) = r \wedge mv$$
$$= r \wedge P = L$$

$L = r \wedge P$ This is the angular momentum or moment of momentum about a reference point O.

$$T = \frac{dL}{dt}$$

The torque acting on a particle equals the time rate of change in its angular momentum.

2.5 Conservation of Momentum

Newton's Second Law $F = \frac{d}{dt}(mv)$

Let F = 0, $\frac{d}{dt}(mv) = 0$

mv = constant

If the net external force acting on a particle is zero, its momentum will remain unchanged. This is the principle of conservation of momentum.

2.51 Conservation of Angular Momentum

If the net external torque acting on a particle is zero, the angular momentum will remain unchanged.

$$T = \frac{dL}{dt} = \frac{d}{dt} m(r \wedge v) = 0$$

$$m(r \wedge v) = \text{constant}$$

Example

Show that the force field F defined by

$F = (y^2 z^3 - 6xz^2)i + 2xyz^3 j + (3xy^2 z^2 - 6x^2 z)k$ is a conservative force field.

Solution

The force field F is conservative if and only if curl F = $\nabla \wedge F = 0$

$$\Delta \wedge F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 - 6xz^2 & 2xyz^3 & 3xy^2 z^2 - 6x^2 z \end{vmatrix}$$

$$= \begin{array}{l} i\left[\dfrac{\partial}{\partial y}\left(3xy^2z^2 - 6x^2z\right) - \dfrac{\partial}{\partial z}\left(2xyz^3\right)\right] \\ + j\left[\dfrac{\partial}{\partial z}\left(y^2z^3 - 6xz^2\right) - \dfrac{\partial}{\partial x}\left(3xy^2z^2 - 6x^2z\right)\right] \\ + k\left[\dfrac{\partial}{\partial x}\left(2xyz^3\right) - \dfrac{\partial}{\partial y}\left(y^2z^3 - 6xz^2\right)\right] \end{array}$$

$$\Delta \wedge F = i\left[6xyz^2 - 6xyz^2\right] + j\left[3y^2z^2 - 12xz - 3y^2z^2 + 12xz\right] + k\left[2yz^3 - 2yz^3\right] = 0$$

$\Delta \wedge F = 0$. Hence, the force field is conservative.

2.6 Central Forces Problems

A force whose direction always passes through a fixed point is called a central force. Let us suppose that a force acting on a particle of mass m such that (a) it is always directed from m toward or away from a fixed point O. (b) its magnitude depends only on the distance r from O. Then we call such a force a central force or central force field with O as the center of force. Symbolically, F is a central force if and only if $F = f(r)r_1 = f(r)\frac{\mathbf{r}}{r}$ where $r_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of r.

The central force is one of attraction toward O or repulsion from O according as f(r) < 0 or f(r) > 0 respectively.

Examples

Scattering of a particle by a central repulsive inverse square force. Consider a particle subject to a repulsive force inversely proportional to the square of the distance from the moving particle to a fixed point or centre of force. [This problem is applicable in atomic and nuclear Physics]. When a proton, accelerated by a machine such as a cyclotron passes near a nucleus of the target material, the proton is deflected or scattered under the action of such a force due to the electric repulsion of the nucleus.

2.61 Important Properties of Central Force Fields

If a particle moves in a central force field, then

1. The path or orbit of the particle must be a plane curve i.e. the particle moves in a plane (x-y plane).
2. The angular momentum of the particle is conserved, that is, is constant.
3. The particle moves in such a way that the position vector or radius vector drawn from O to the particle sweeps out equal areas in equal times. In other words, the time rate of change in area is constant.

2.62 Equations of Motion for a Particle in a Central field

We know that the motion of a particle in a central force field takes place in a plane. If we choose this plane as the x-y plane and the coordinates of the particle as polar coordinates (r,θ), the equation of motion are found to be

$$m\left(\ddot{r} - r\dot{\theta}\right) = f(r)$$

$$m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = 0$$

dots denote differentiations with respect to time t.

2.63 Potential Energy of A Particle in A central Field

$$V(r) = -\int f(r)dr$$

This is the potential energy of a particle in the central force field. We obtain the additive constant by assuming $V = 0$ at r = 0 or $V \to 0$ as $r \to \infty$.

2.64 Conservation of Energy

Using the last equation and the fact that in polar coordinates the kinetic energy of a particle is $\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$, the equation for conservation of energy can be written $\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E$

Or

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \int f(r)dr = E$$ E is the total energy and is constant.

CHAPTER THREE

3.0 Oscillations

3.1 Periodic Motion

This is any type of motion that keeps on repeating itself after a certain interval of time T known as the period of the motion. For example, the motion of planets around the sun, the motion of an oscillating pendulum.

3.11 Simple Harmonic Motion

This is an example of periodic motion. It is defined as the motion of a body in a straight line whose acceleration is directed to a fixed point and is directly proportional (in magnitude) to the distance of the body from the point.

$$|a| \propto x$$

$$|a| = cx$$

$$a = -cx$$

c is a constant. The acceleration is pointing in the opposite direction of the motion.

$$a = \frac{d^2x}{dt^2} = -cx \text{ motion along x – axis}$$

$$\frac{d^2x}{dt^2} + cx = 0$$

From Hooke’s law, the equation of motion of the body in differential from will be

$$m\frac{d^2x}{dt^2} = -kx$$

$$m\frac{d^2x}{dt^2k} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This can be written as $\frac{d^2x}{dt^2} + \omega^2 x = 0$, $\omega^2 = \frac{k}{m}$, ω is known as the angular frequency of the motion. $\omega = 2\pi f$, f is the frequency of the motion.

The solution of the last equation gives us the instantaneous displacement of the body from the equilibrium. The solution is

$$x(t) = ACos\omega t + B\sin\omega t$$

$x(t) = aCos(\omega t + \phi)$, where a and ϕ are constant.

3.12 Instantaneous Velocity and Acceleration of Simple Harmonic Motion

$$x(t) = aCos(\omega t + \phi)$$

$$v(t) = \frac{dx(t)}{dt} = -a\omega Sin(\omega t + \phi)$$

if $\phi = 0$, $x = aCos\phi$

$$Cos\phi = \frac{x}{a}$$

$$Sin\phi = \sqrt{1 - Cos^2\phi}$$

$$Sin\phi = \sqrt{1 - \frac{x^2}{a^2}}$$

$$v(t) = -a\omega Sin(\omega t + \phi)$$

$$v(t) = -a\omega\sqrt{1 - \frac{x^2}{a^2}}$$

$$v(t) = -\omega\sqrt{a^2 - x^2}$$

minus indicates that the direction of velocity is always opposing the motion.

At extreme ends, v = 0 because x = a, v = v_{max} at x = 0, $v = -\omega a$

$$a(t) = \frac{dv}{dt} = -a\omega^2 Cos(\omega t + \phi)$$

$$a(t) = -\omega^2 x$$

The acceleration is also opposing the motion. a = a_{max} at x = a

3.13 Energy of A Simple Harmonic Motion (or Oscillator)

In a S.H.M. there are no dissipative forces, hence, the total energy is equal to the mechanical energy of the system, that is, the sum of potential energy and the kinetic energy.

$$E = T + V$$

$$T = \frac{1}{2}mv^2 \quad \text{and} \quad V = \frac{1}{2}kx^2$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

3.2 Damped Oscillations

A physical system undergoing SHM is an ideal case since in theory the oscillations will continue forever with constant amplitude, frequency and phase angle. What we normally experience in real life are oscillations whose amplitudes die out gradually to zero. The reason is that there are external resistant forces on the motion of such bodies. Hence, there are energy losses. This type of oscillation taking place in the absence of an external periodic force aid, the oscillation is known as damped oscillation. The nature of the damping depends on the manner in which the damping agent resists the oscillating body. We shall consider a simple case where the damping is proportional in magnitude to only the first power of the velocity. That is,

$$F_D \propto \frac{dx}{dt} \quad \text{for oscillation along the x – axis.}$$

$$F_D = -k' \frac{dx}{dt}$$

Examples

1. When a mass attached to an elastic spring is vibrating inside a viscous liquid.

2. When a simple (or compound) pendulum is oscillating in a windy environment.

For such oscillations in the x – direction the equation of motion in differential form becomes

$$M\frac{d^2x}{dt^2} = -kx - k'\frac{dx}{dt}$$ where k' is known as the coefficient of damping.

$$M\frac{d^2x}{dt^2}+k^1\frac{dx}{dt}+kx=0$$

$$\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}+\omega^2x=0$$

$$\alpha=\frac{k^{'}}{m}\text{ and }\omega^2=\frac{k}{m},$$

The behaviour of the system (that is, the nature of vibration) depends on the term

$$\Delta=\frac{\alpha^2}{4}-\omega^2$$

3.21 Under Damped Motion

Here

$$\Delta<0\Rightarrow\frac{\alpha^2}{4}<\omega^2$$

$$\Rightarrow\frac{k^{'2}}{4m^2}<\frac{k^2}{m}$$

The motion is said to be under damped. The solution is of the form

$$x(t)=Ae^{-\frac{\alpha}{2}t+i\sqrt{\omega^2-\frac{\alpha^2}{4}t}}+Be^{-\frac{\alpha}{2}t-i\sqrt{\omega^2-\frac{\alpha^2}{4}t}}$$

3.22 Critically Damped Motion

Here $\Delta = 0$.

$$\Rightarrow\frac{\alpha^2}{4}=\omega^2$$

$$\Rightarrow\left(\frac{k^{'}}{2m}\right)^2=\frac{k}{m}$$

The system is said to be critically damped under this condition. The solution should be of the form

$x(t) = Pe^{-\frac{\alpha}{2}t} + Qte^{-\frac{\alpha}{2}t}$ where P and Q are constants whose values depend on the initial conditions. Here, there are no vibrations, the mass only returns to the equilibrium position after it has been displaced and released.

3.23 Over Damped Motion

This occurs when $\Delta > 0$, that is, $\frac{\alpha^2}{4} > \omega^2 \Rightarrow \left(\frac{k'^2}{2m}\right)^2 > \frac{k}{m}$

Here, the solution is

$$x(t) = \mathrm{R}e^{-\frac{\alpha}{2}t + \sqrt{\frac{\alpha^2}{4} - \omega^2}t} + Se^{-\frac{\alpha}{2}t - \sqrt{\frac{\alpha^2}{4} - \omega^2}t}$$

R and S are arbitrary constants. No vibration here as well, the mass returns to the equilibrium position after being displaced and released, but it does so more slowly than in a critically damped case.

3.3 Forced Vibrations or Oscillations

Forced vibrations takes place when a system capable of vibrating is subjected to an external force which is itself periodic. In this case, the behaviour of the system depends on the nature of the external periodic force applied.

Let's consider the case where a mechanical system is undergoing vibrations in a resisting medium and is also subjected to an external force which is varying simple harmonically as $F = F_o Cosbt$.

The equation of motion here, for vibration along the x – axis is

$$m\frac{d^2x}{dt^2} + k'\frac{dx}{dt} + kx = F_o Cosbt$$

This is second order Linear Differential Equation and it is Non – homogenous. It has a general solution of the form

$X_G = X_C + X_P$ Method of undetermined coefficient where X_C is the complementary solution, X_P is particular solution and X_G is the general solution.

CHAPTER FOUR

4.0 Collision of Particles, Moving Frames of Reference and Elementary Mechanics of Rigid Bodies

4.1Collision of Particles

Two or more particles may collide with each other during the course of their motions. Problems which consider the motions of such particles are called collision or impact problems.

In practice we think of colliding objects such as spheres, as having elasticity. The time during which such objects are in contact is composed of a compression time during which slight deformation may take place, and restitution time during which the shape is restored. We assume that the spheres are smooth so that forces exerted are along the common normal to the spheres through the point of contact (and passing through their centers).

A collision can be direct or oblique. In a direct collision the direction of motion of both spheres is along the common normal at the point of contact both before and after collision. A collision which is not direct is called oblique.

4.11 Newton's Collision Rule

If v_{12} and $v_{12}^{'}$ are the relative velocities of the spheres along the common normal before and after impact. Then $v_{12}^{'} = -\varepsilon v_{12}$. The quantity ε, is the coefficient of restitution and it depends on the materials of which the objects are made and is generally taken as a constant between 0 and 1. If $\varepsilon = 0$ the collision is called perfectly inelastic or briefly inelastic. If $\varepsilon = 1$ the collision is called perfectly elastic or briefly elastic. In this case, the total kinetic energy before and after impact is the same.

Example

Two masses m_1 and m_2 travelling in a straight line collide. Find the velocities of the particles after collision in terms of the velocities before collision.

Solution

Let the velocities of the particles before and after collisions be v_1, v_2 and $v_1^{'}, v_2^{'}$ respectively. By Newton's Collision Rule $v_1^{'} - v_2^{'} = \varepsilon(v_2 - v_1)$ 1

By the principle of conservation of momentum,

total momentum after collision = total momentum before collision

$$m_1 v_1^{'} + m_2 v_2^{'} = m_1 v_1 + m_2 v_2 \qquad 2$$

Solving equations 1 and 2 simultaneously

$$v_1' = \frac{(m_1 - \varepsilon m_2)v_1 + m_2(1+\varepsilon)v_2}{m_1 + m_2}$$

$$v_2' = \frac{m_1(1+\varepsilon)v_1 + (m_2 - \varepsilon m_1)v_2}{m_1 + m_2}$$

4.2 Moving Frames of Reference

The coordinates systems used to describe the motions of particles by Newton were assumed to be inertial frame of reference, that is, they are fixed in space, that is, is absolutely at rest. In practice, this assumption will cause to hold, for example, a coordinate system fixed in the earth is not an inertial system since the earth itself is rotating in space. Consequently, if we use this coordinate system to describe the motion of a particle relative to the earth we obtain results which may be in error. Therefore, it is better to consider the motion of particles relative to moving coordinate system. This moving coordinate system is referred to as non-inertial frame of reference.

4.21 Rotating Coordinate

If XYZ denote an inertial coordinate system with origin O which is considered fixed in space. Let the coordinate system xyz having the same origin O be rotating with respect to the XYZ system. Let us consider vector A which is changing with time. To an observer fixed relative to the xyz system the time rate of change of $A = A_1 i + A_2 j + A_3 k$ is found to be

$$\left.\frac{dA}{dt}\right|_M = \frac{dA_1}{dt}i + \frac{dA_2}{dt}j + \frac{dA_3}{t}k \qquad 1$$

where subscript M indicates the derivative in the moving (xyz) system. However, the time rate of change of A relative to the fixed XYZ system symbolized by the subscript F is found to be

$$\left.\frac{dA}{dt}\right|_F = \left.\frac{dA}{dt}\right|_M + \omega \wedge A \qquad 2$$

ω is called the angular velocity of the xyz system with respect to the XYZ system.

Let D_F and D_M represent time derivative operators in the fixed and moving systems. Then we can write operator equivalence

$$D_F \equiv D_M + \omega \wedge \qquad 3$$

4.22 Velocity in A Moving System

If, in particular, vector A is the position vector r of a particle, then equation 2 gives

$$\left.\frac{dr}{dt}\right|_F = \left.\frac{dr}{dt}\right|_M + \omega \wedge r \qquad 4$$

$$D_F r = D_M r + \omega \wedge r \qquad 5$$

or

$v_{P/F} = \left.\frac{dr}{dt}\right|_F = D_F r$ = velocity of particle P relative to fixed system

$v_{P/M} = \left.\frac{dr}{dt}\right|_M = D_M r$ = velocity of particle P relative to moving system

$v_{M/F} = \omega \wedge r$ = velocity of moving system relative to fixed system.

Then, equations 4 and 5 can be written as

$$v_{P/F} = v_{P/M} + v_{M/F} \qquad 6$$

4.23 Acceleration in A Moving System

If $D_F^2 = \left.\frac{d^2}{dt^2}\right|_F$ and $D_M^2 = \left.\frac{d^2}{dt^2}\right|_M$ are second derivative operators with respect to t in the fixed and moving systems, then application of equation 3 yields

$$D_F^2 r = D_M^2 r + (D_M \omega) \wedge r + 2\omega \wedge D_M r + \omega \wedge (\omega \wedge r) \qquad 7$$

4.24 Coriolis and Centripetal Acceleration

The last two terms on the right of equation 7 are called the Coriolis acceleration and Centripetal acceleration respectively.

Coriolis acceleration = $2\omega \wedge D_M r$

= $2\omega \wedge v_M$

Cenripetal acceleration = $\omega \wedge (\omega \wedge r)$

Example

An xyz coordinate system is rotating with respect to an XYZ coordinate system having the same origin and assumed to be fixed in space. The angular velocity of the xyz system relative to the XYZ system is given by $\omega = 2ti - t^2 j + (2t+4)k$ where t is time. The position vector of a particle at time t as observed in the xyz system is given by $r = (t^2 + 1)i - 6tj + 4t^3 k$. Find (a) the apparent velocity and (b) the true velocity at time t = 1.

Solution

(a) The apparent velocity at any time t is $\frac{dr}{dt} = 2ti - 6j + 12t^2 k$

At time t = 1, this is $\frac{dr}{dt} = 2i - 6j + 12k$

(b) The true velocity at any time t is

$$\frac{dr}{dt} + \omega \wedge r = \left(2ti - 6j + 12t^2 k\right) + \left[2ti - t^2 j + (2t+4)k\right] \wedge \left[(t^2 + 1)i - 6tj + 4t^3 k\right]$$

At time t = 1, this is

$$2i - 6j + 12k \quad + \quad \begin{vmatrix} i & j & k \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix} = 34i - 2j + 2k$$

4.3 Elementary Mechanics of Rigid Bodies

4.3.1 What is A Rigid Body?

This is a system of particles in which the distance between any two particles does not change regardless of the forces acting on it.

4.3.2 Translations and Rotation

A displacement of a rigid body is a change from one position to another. If during a displacement all points of the body on some line remain fixed, the displacement is called a Rotation about the line. If during a displacement all points of the rigid move in lines parallel to each other the displacement is called a Translation.

4.3.3 Euler's Theorem

A rotation of a rigid body about a fixed point of the body is equivalent to a rotation about a line which passes through the point. The line referred to is called the instantaneous axis of rotation.

4.3.4 Chasle's Theorem

The general motion of a rigid body can be considered as a translation plus a rotation about a suitable point which is often taken to be the center of mass.

4.3.5 Plane Motion of A Rigid Body

The motion of a rigid body is simplified considerably when all points move parallel to a given fixed plane. In such case two types of motion, called plane motion are possible.

4.3.5.1 Rotation About A Fixed Axis

In this case the rigid body rotates about a fixed axis perpendicular to the fixed plane. The system has only one degree of freedom and thus only one coordinate is required for describing the motion.

4.3.5.2 General Plane Motion

In this case, the motion can be considered as a translation parallel to the given fixed plane plus a rotation about a suitable axis perpendicular to the plane. This axis is often chosen so as to pass through the center of mass. The number of degrees of freedom for such motion is three (3): two coordinates being used to describe the translation and one to describe the rotation. The axis referred to is the instantaneous axis and the point where the instantaneous axis intersects the fixed plane is called the instantaneous center of rotation.

4.3.6 Moments of Inertia

The moment of inertia of a particle of mass m about a line or axis say AB is defined as $I = mr^{2}$, where r is the distance from the mass to the line. The moment of inertia of a system of particles, with masses m_1, m_2, ……m_N about the line or axis AB is defined as

$$I = \sum_{v=1}^{N} m_v r_v^2 = m_1 r_1^2 + m_2 r_2^2 + ... + m_N r_N^2$$

r^1, r^2, ….. r^N are their respective distances from AB.

The moment of inertia of a continuous distribution of mass, such as the solid rigid body R is given by

$$I = \int_R r^2 dm$$

r is the distance of the element of mass dm from AB.

4.3.7 Radius of Gyration

If $I = \sum_{v=1}^{N} m_v r_v^2$ is the moment of inertia of a system of particles about AB and

$M = \sum_{v=1}^{N} m_v$ is the total mass of the system.

Then, the quantity K such that

$$K^2 = \frac{I}{M} = \frac{\sum_v m_v r_v^2}{\sum_v m_v}$$

is called the radius of gyration of the system about AB.

For continuous mass distribution, we have

$$K^2 = \frac{I}{M} = \frac{\int_R r^2 dm}{\int_R dm}$$

4.3.8 Theorems

4.3.8.1 Parallel Axis Theorem

Let I be the moment of inertia of a system about axis AB and let I_C be the moment of inertia of the system about an axis parallel to AB and passing through the center of mass of the system. Then if b is the distance between the axes and M is the total mass of the system, we have

$$I = I_C + Mb^2$$

4.3.8.2 Perpendicular Axis Theorem

Consider a mass distribution in the x-y plane of an xyz coordinate system. Let I_x, I_y and I_z denote the moments of inertia about the x, y and z axes respectively. Then,

$$I_z = I_x + I_y$$