COURSE CODE: **MTS 103**

COURSE TITLE: **VECTORS AND GEOMETRY**

NUMBER OF UNITS: **3 UNITS**

COURSE DURATION: **THREE HOURS PER WEEK**

COURSE COORDINATOR: **DR. B. I OLAJUWON**

OTHER LECTURER **DR. O. T MEWOMO, MR. E. ILOJIDE**

COURSE REQUIRMENT:

This course is compulsory course for for all students in 100 level Mathematics, Physics, Computer sciences, Statistics and College of Engineering. In view of this, students are expected to participate in all the course activities and have minimum of 75% attendance to be able to write the final examination.

COURSE CONTENT:

Equations of straight lines, intersecting and perpendicular lines, equations of lines and planes. Conic sections; circle, parabola, hyperbola and ellipse. Geometric representation of vectors in R2 and R3, components of direction cosines. Addition, scalar multiplication of vectors, linear independence. Scalar vector product of two vectors. General equation of a conic in polar coordinates.

READING LIST:

1. Tuttuh – Adegun, M. R, Sivassubramaniam, Adegoke, R, Further Mathematics Project, NPS Educational Publisher Limited, Ibadan, Nigeria, 1992.
2. A. P Armit, Advanced level vectors, Heinemann Educational Books London, 2nd Edition, 1985.

1.0 The Straight Line

1.1 The Equation of the Straight Line

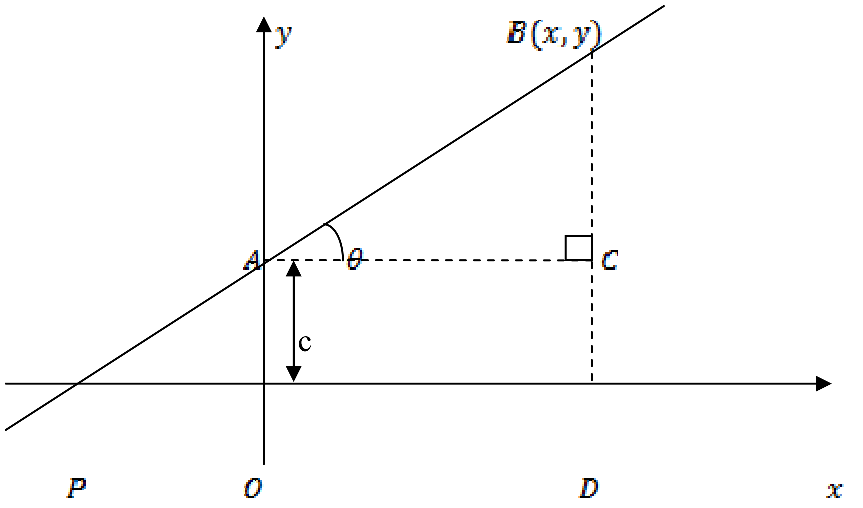


Figure 1

Consider figure 1 in which PAB is the straight line, angle $APO = \theta, OA = c, B(x, y)$ is any point on the straight line, BD is the ordinate and AC is parallel to Ox.

It is clear that

$$\angle BAC = \angle APO \qquad \text{(Corresponding angles)}$$

Therefore,

$$\tan\theta = \frac{BC}{AC}$$

$$\tan\theta = \frac{y - c}{x}$$

$$y = x\tan\theta + c,$$

$\tan\theta$ is the slope of the straight line denoted by m. Thus $y = mx + c$ is the gradient intercept form of the straight line.

1.2 Other Forms of the Equation of a Straight Line

Gradient and One Point Form

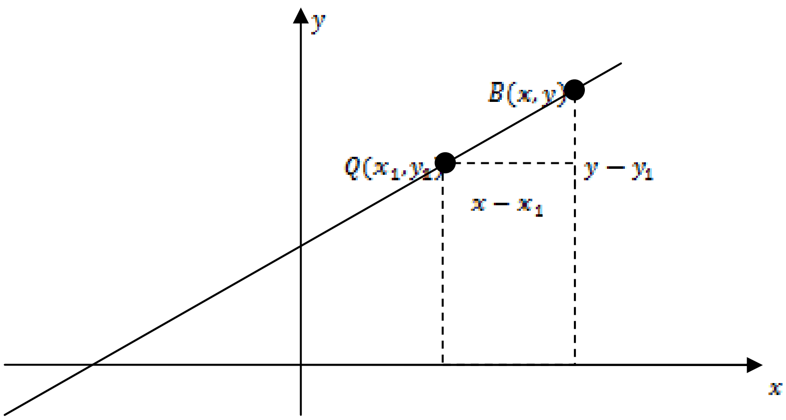


Figure 2

Consider figure 2 in which a straight line is passing through $Q(x, y)$ and having a gradient m. $B(x, y)$ is a variable point on the straight line, thus

$$\text{Gradient of } QB = \frac{y - y_1}{x - x_1}$$

$$m = \frac{y - y_1}{x - x_1}$$

Or $y - y_1 = m(x - x_1)$

Example: Find the equation of a straight line of slope 3, if it passes through the point $(-2, 3)$.

Solution: The equation of a straight line of gradient m passing through (x_1, y_1) is

$y - y_1 = m(x - x_1)$

$m = 3, x_1 = -2, y_1 = 3$

Hence the equation of the straight line is

$y - 3 = 3(x — 2)$

$y - 3 = 3(x + 2)$

$y = 3x + 9$

Two Points Form

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a straight line with slope m. We take a variable point $B(x, y)$ on the line, as shown below.

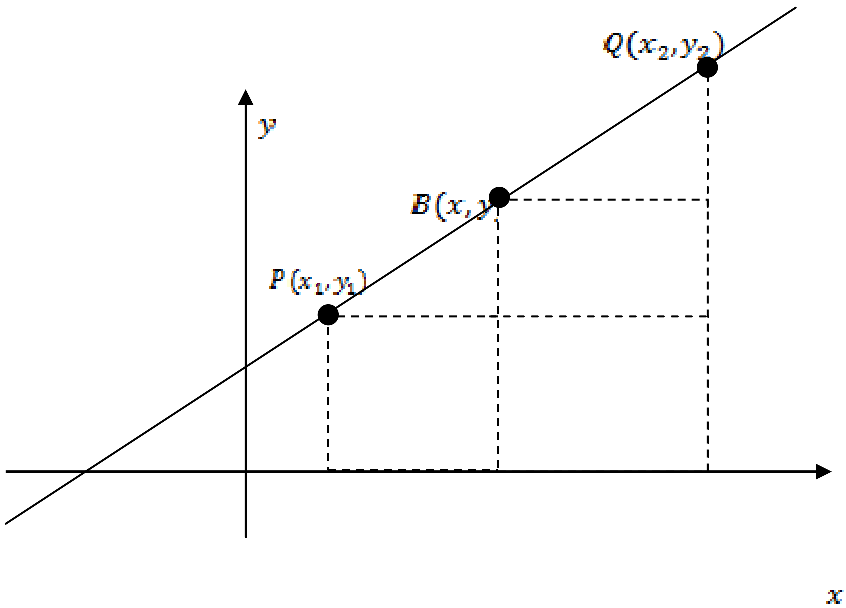


Figure 3

From figure 3,

$$\text{Gradient } PB = \frac{y - y_1}{x - x_1}$$

$$\text{Gradient } PQ = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus the equation of the straight line is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the equation of the straight line which passes through the points $A(2,-3)$ and $B(3,2)$.

Solution

$(x_1,y_1) = (2,-3), (x_2,y_2) = (3,2)$

Hence the equation of the straight line through A and B is

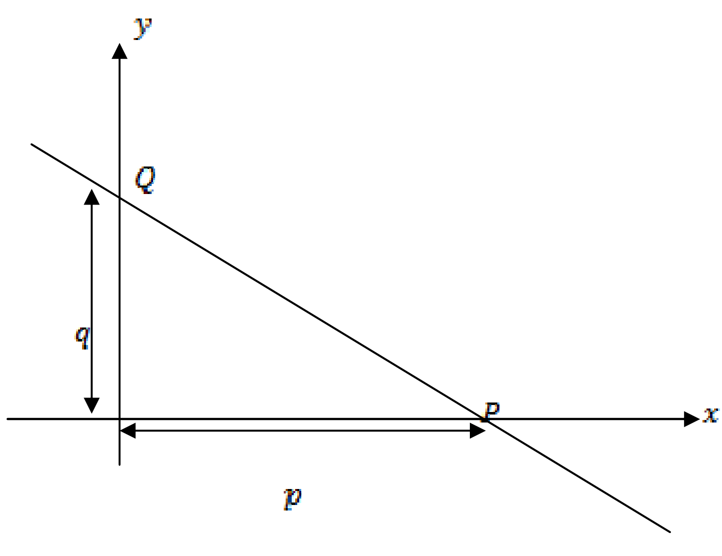
$$\frac{y-(-3)}{x-2} = \frac{2-(-3)}{3-2}$$

$$\frac{y+3}{x-2} = \frac{2+3}{1}$$

$$y+3 = 5x-10$$

$$y = 5x-13$$

The Equation of the Straight Line making Intercepts p and q on Ox and Oy respectively



Let the straight line cross the x and y axes at P and Q respectively. Let the coordinate at P be $(p, 0)$ and at Q be $(0, q)$. Thus, using the two point form formula,

$$\frac{y-q}{x} = \frac{0-q}{p}$$

$$py - pq = -xq$$

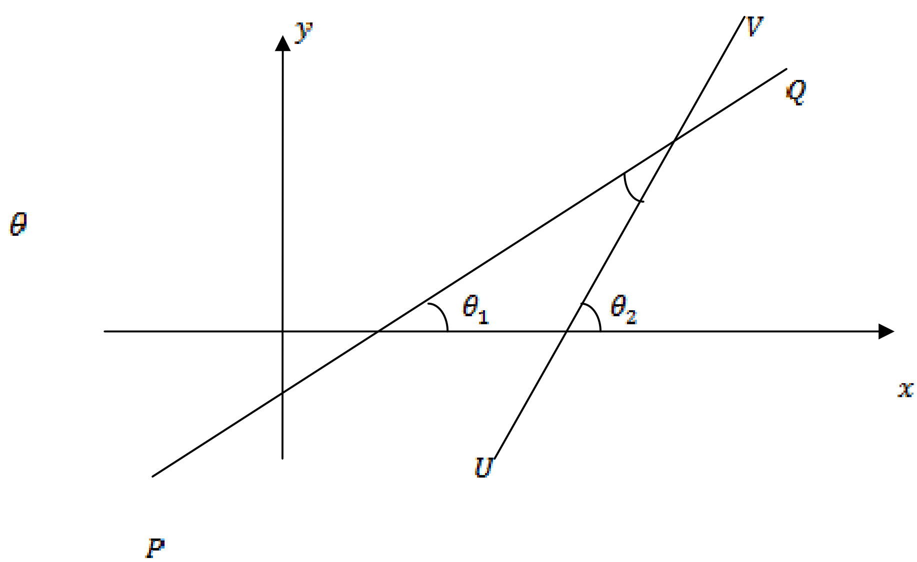
$$py + xq = pq$$

$$\therefore \ \frac{y}{q} + \frac{x}{p} = 1$$

This is called the intercept form of the equation of the straight line.

Other forms of equation of a straight line exist. The reader should endeavour to find out.

1.3 The Angle Between Two Straight Lines



Let PQ and UV be two given straight lines (as shown in the figure above) with gradients m_1 and m_2 respectively. The acute angle between the lines is θ and $\theta = \theta_2 - \theta_1$. From our previous knowledge $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$. Thus

$\tan\theta = \tan(\theta_2 - \theta_1)$

$= \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$

$= \dfrac{m_2 - m_1}{1 + m_1 m_2}$

Hence $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is negative, we obtain the obtuse angle $180 - \theta$. The acute angle θ between the two lines is given by

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$

Remark

1. If $\tan\theta = 0$, the two lines are parallel. Then $m_2 = m_1$.
2. If the lines are perpendicular to each other, then $\theta = 90^o$ and $\tan\theta = \infty$.

Therefore,

$1 + m_1 m_2 = 0$

$m_1 m_2 = -1$

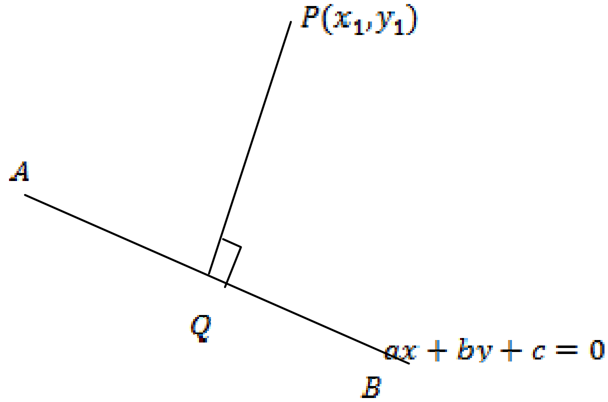
$m_1 = \dfrac{-1}{m_2}$

1.4 The Perpendicular Distance of a Point from a Straight Line

Let $P(x_1, y_1)$ be the point and the equation of the line be

$$ax + by + c = 0 \quad (1)$$

as shown below.



Let the straight line $ax + by + c = 0$ be denoted AB and the line perpendicular to it be PQ. From the equation $ax + by + c = 0$

$$y = \frac{-a}{b}x - \frac{c}{b}$$

Thus, the gradient of line AB is $\frac{-a}{b}$. Therefore the gradient of the line PQ is $\frac{b}{a}$.

Using the point formula, the equation of the straight line PQ is derived thus,

$$\frac{b}{a} = \frac{y - y_1}{x - x_1}$$

Hence, the equation of the line PQ is

$$-bx + ay = ay_1 - bx_1 \quad (2)$$

Solving (1) and (2) simultaneously to determine the coordinate of $Q(x_2, y_2)$, we have that

$$x_2 = \frac{b^2 x_1 - aby_1 - ac}{a^2 + b^2} \text{ and}$$

$$y_2 = \frac{a^2 y_1 - abx_1 - bc}{a^2 + b^2}$$

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2} - x_1\right)^2 + \left(\frac{a^2y_1 - abx_1 - bc}{a^2 + b^2} - y_1\right)^2}$$

$$= \sqrt{\frac{a^2(ax_1 + by_1 + c)^2 + b^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}}$$

$$= \sqrt{\frac{(a^2 + b^2)(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}}$$

$$= \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

is the length of the perpendicular.

Example: Find the length of the perpendicular from the point $P(2, -4)$ to the line $3x + 2y - 5 = 0$.

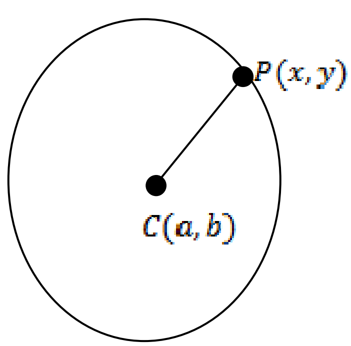
Solution

The length is $\frac{3(2) + 2(-4) - 5}{\sqrt{3^2 + 2^2}} = \frac{6 - 8 - 5}{\sqrt{13}} = \frac{-7}{\sqrt{13}}$

Hence the perpendicular distance is $\frac{7}{\sqrt{13}}$.

2.0 The Equation of a Circle

Consider a circle with centre $C(a, b)$ and radius r (as shown below). Let $P(x, y)$ be an arbitrary point on the circumference of the circle.



It follows that

$CP = r$

$CP^2 = r^2$

Using the expression for the distance between two points, we have

$$(x-a)^2 + (y-b)^2 = r^2 \tag{2.1}$$

Or

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2 \tag{2.2}$$

If we let $a = b = 0$, the centre of the circle is the origin and the equation reduces to

$$x^2 + y^2 = r^2 \tag{2.3}$$

If we let $a = -g, b = -f$ and $r = \sqrt{g^2 + f^2 - c}$, the equation of a circle is thus of the form

$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

This implies that the circle has centre $(-g, -f)$.

Example

Find the equation of the circle with centre $(3,7)$ radius 5.

Solution

The equation is:

$(x-3)^2 + (y-7)^2 = 5^2$

$x^2 - 6x + 9 + y^2 - 14y + 49 = 25$

$x^2 + y^2 - 6x - 14y + 33 = 0$

Exercise

Find the centre and radius of the circle $4x^2 + 4y^2 - 12x + 5 = 0$.

2.2 The of a Circle through Three Non-collinear Points

Let the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be points on the circumference of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Since the circle passes through all the three points, the coordinates of each point must satisfy the equation of the circle.

Hence

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

Solving these equations simultaneously we obtain the values of g, f and c.

Example: Find the equation of the circle through the points $(0,0), (3,1)$ and $(5,5)$ and determine the radius.

Solution

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Then since $(0,0)$ lies on the circle, $c = 0$.

Similarly,

$$9 + 1 + 6g + 2f = 0$$

and

$$25 + 25 + 10g + 10f = 0$$

Solving these simultaneous equations, we have $f = -5, g = 0$.

Hence the required equation is:

$x^2 + y^2 - 10y = 0$

2.3 The Equation of the Tangent at the Point (x_1, y_1) on a Circle

Let $P(x_1, y_1)$ be a point on the circumference of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

We need to determine the equation of the tangent to the circle at the point (x_1, y_1).

Differentiating the equation with respect to x, we have

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0,$$

Therefore

$$\frac{dy}{dx} = \frac{-(x+g)}{(y+f)}$$

is the gradient of the tangent at the point (x_1, y_1). Thus the equation of the tangent is

$$\frac{-(x+g)}{(y+f)} = \frac{y-y_1}{x-x_1}$$

$$xx_1 + yy_1 + gx + fy = x_1{}^2 + y_1{}^2 + gx_1 + fy_1$$

Adding $gx_1 + fy_1 + c$ to both sides yields

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1{}^2 + y_1{}^2 + 2gx_1 + 2fy_1 + c = 0$$

Since (x_1, y_1) lies on the circle. Hence the required equation is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Exercises

(i) Given the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, determine the length of the tangent from a point $P(X, Y)$ outside the circle.

(ii) Given the circle $x^2 + y^2 = r^2$ and the straight line $y = mx + c$, determine their points of intersection.

Lecture Guide on Elementary Vector Operations

Definition

Scalars and vectors: Scalar is a single real number called magnitude and is not related to any direction in space. The vector is a quantity which has a magnitude as well as a definite direction in space. The speed of a bus is a scalar quantity but the velocity is a vector quantity.

Line vectors: The vector $\overline{AB}$, A is called the origin and B the terminus. The magnitude of the vector is given by the length AB and its direction is from A to B. These vectors are called line vectors.

Equal vectors: Two vectors are said to be equal when they have the] same length (magnitude) and are parallel having the same sense of direction. The equality of two vectors is written as $\overline{a} = \overline{b}$.

Zero vectors: If the origin and terminal points of a vector are same, then it is said to be a zero vector. Evidently its length is zero and its direction is indeterminate.

Unit vector: A vector is said to be a unit vector if its magnitude be of unit length.

Position vector: The position vector of any point P, with reference to an origin O is the vector $\overline{OP}$. Thus taking O as origin we can find the position vector of every point in space. Conversely, corresponding to any given vector $\overline{r}$ there is a point P such that $\overline{OP} = \overline{r}$

Addition of two vectors: Let $\bar{a}$ and **Error! Objects cannot be created from editing field codes.** be two vectors with respect to the origin O. The sum of these two vectors is given by $\bar{a}+\bar{b}$.

The unit vectors $\bar{i}, \ \bar{j}, \ \bar{k}$ **:** The vectors $\bar{i}, \ \bar{j}, \ \bar{k}$ have unit magnitude and they lie on the *x, y* and *z* axes respectively. We can expresses any vector in terms of these three unit vectors $\bar{i}, \ \bar{j}, \ \bar{k}$.

Collinear vectors: Two vectors $\bar{a}$ and **Error! Objects cannot be created from editing field codes.** are said to be collinear if $\bar{a}=\lambda\bar{b}$, for some scalar λ, i.e., two vectors are collinear if the coefficient of $\bar{i}, \ \bar{j}$ and $\bar{k}$ are proportional.

Magnitude of a vector: Let $\bar{a}=a_1 i+a_2 j+a_3 k$. Then the magnitude or length of the vector $\bar{a}$ is denoted by $|\bar{a}|$ or a and is defined as $|\bar{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$.

Distance between two points: Let P_1 and P_2 be two points whose position vectors are respectively $\bar{a}=a_1 i+a_2 j+a_3 k$ and $\bar{b}=b_1 i+b_2 j+b_3 k$. Then the vector $\overline{P_1P_2}$ = position vector of P_2 - position vector of $P_1=(b_1 i+b_2 j+b_3 k)-(a_1 i+a_2 j+a_3 k)$

Then the distance between two points P_1 and P_2 is the magnitude of the vector $\overline{P_1P_2}$.

*** The unit vectors in the direction of a vector a are given as $\pm\frac{\bar{a}}{|\bar{a}|}$.

Examples:

1. Find the value of q if $\bar{a}=2i+5j+qk$. If the magnitude of $\bar{a}$ is 9

2. Given vectors $\bar{a} = 5i + j + 3k$, $\bar{b} = i - 3j + 4k$ and $\bar{c} = 7i + 2j - 3k$. find the unit vector in the direction of $\bar{a} - \bar{b} + 2\bar{c}$.

3. Find the distance between A and B whose-position vectors are $\bar{a} = 5i + j + 3k$, $\bar{b} = i - 3j + 4k$ respectively.

4. Prove by vector method that the three points *A (2,* 3, 4), *B (1,* 2, 3) and

C (4, 2, 3) form a right-angled triangle.

5. Show that the three points *-3i – 6j + 21k, 9i +3k and 15i + 3j – 6k* are collinear.

Scalar Product or Dot Product

The scalar or dot product between vectors $\bar{a}$ and **Error! Objects cannot be created from editing field codes.** is denoted by $\bar{a} \bullet \bar{b}$ and defined as $\bar{a} \bullet \bar{b} = |\bar{a}||\bar{b}|\cos\theta$, where θ is the angle between $\bar{a}$ and **Error! Objects cannot be created from editing field codes.**. The value of $\bar{a} \bullet \bar{b}$ is a scalar quantity.

*** Two vectors $\bar{a}$ and **Error! Objects cannot be created from editing field codes.** are perpendicular if and only if $\bar{a} \bullet \bar{b} = 0$

Examples:

1. Given vectors $\overline{a} = 5i + j + 3k$, $\overline{b} = i - 3j + 4k$ and $\overline{c} = 7i + 2j - 3k$.

Find (i) $\overline{a} \bullet \overline{b}$ (ii) $(\overline{a} \bullet \overline{b}) \bullet \overline{c}$ (iii) angle between $\overline{a} - \overline{c} + 2\overline{b}$ and $\overline{a} + \overline{b} + \overline{c}$

2. Find the value of q for which the two vectors are $\overline{a} = 5i + qj + 3k$ and $\overline{b} = i - 3j + 4k$ are perpendicular to each other.

Vector Product or Cross Product

The vector product or cross product between two vectors $\overline{a}$ and **Error! Objects cannot be created from editing field codes.** is denoted by $\overline{a} \times \overline{b}$ and is defined by $\overline{a} \times \overline{b} = |\overline{a}||\overline{b}| \sin \theta$, where θ is the angle between $\overline{a}$ and **Error! Objects cannot be created from editing field codes..**

NOTE:

(i) $\overline{a} \times \overline{b} = -\overline{b} \times \overline{a}$

(ii) If $\overline{a} \times \overline{b} = 0$, then $\overline{a}$ and **Error! Objects cannot be created from editing field codes.** are parallel or collinear

(iii) $\overline{a} \times \overline{a} = 0$

(iv) If $\overline{a}$ and **Error! Objects cannot be created from editing field codes.** represent the adjacent sides of a parallelogram then its area is $|\overline{a} \times \overline{b}|$.

(v) If $\overline{a}$ and **Error! Objects cannot be created from editing field codes.** represent any two sides of a triangle then its area is $\frac{1}{2}|\overline{a} \times \overline{b}|$

(vi) If $\overline{a} \times \overline{b} = \overline{c}$ then $\overline{c}$ is perpendicular to both $\overline{a}$ and **Error! Objects cannot be created from editing field codes.**.

(vii) $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \bullet \overline{c})\overline{b} - (\overline{a} \bullet \overline{b})\overline{c}$

(viii) Three vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are said to be coplanar, if $\overline{a} \bullet (\overline{b} \times \overline{c}) = 0$.

Examples

1. If $\overline{a} = 5i + j + 3k$ and $\overline{b} = i - 3j + 4k$. Find $\overline{a} \times \overline{b}$

2. If $\overline{a}$ and **Error! Objects cannot be created from editing field codes.** are two vectors such that $|\overline{a}| = 16$, $|\overline{b}| = 12$ and $\overline{a} \bullet \overline{b} = 0$ Find $|\overline{a} \times \overline{b}|$.

3. Find the area of the triangle two of whose sides are given by the vectors $\overline{a} = 5i + j + 3k$ and $\overline{b} = i - 3j + 4k$.

4. Find the area of the parallelogram formed by two vectors $\overline{a} = 5i + j + 3k$ and $\overline{b} = i - 3j + 4k$.

5. Find the unit vector perpendicular to each of the vectors $\overline{a} = 5i + j + 3k$ and $\overline{b} = i - 3j + 4k$.

6. Find a vector of magnitude 9 perpendicular to both the vectors $\overline{a} = 5i + j + 3k$ and $\overline{b} = i - 3j + 4k$.

7. Show 'that the vectors $\overline{a} = 4i + 2j + k$, $\overline{b} = 2i - j + 3k$ and $\overline{c} = 8i + 7k$ are coplanar.

8. If a force given by $\overline{F} = 5i + j + 3k$ displaces a particle from the position B to C whose position vectors are $\overline{b} = i - 3j + 4k$ and $\overline{c} = 7i + 2j - 3k$ respectively. Find the work done by the force.

- ***B. I OLAJUWON, 2011***