University of Agriculture, Abeokuta,

Department of Mathematics

2009/2010 Second Semester BSc. Degree Examination October 2010

MTS322 - Vectors and Tensors

INSTRUCTION: Answer Any Four Questions Time:2 Hours

1. (a) (i) let A, B, C and D be vectors such that

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B). \tag{1.1}$$

Determine $P_1 \times (P_2 \times P_3) + P_2 \times (P_3 \times P_1) + P_3 \times (P_1 \times P_2)$, where $P_i (i = 1, 2, 3)$ are vectors.

(ii) If in addition to (1.1), $A \cdot (B \times C) = (A \times B) \cdot C$, show that

$$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C).$$

Hence determine the value of

$$(A \times B) \cdot (C \times D) + (B \times C) \cdot (A \times D) + (C \times A) \cdot (B \times D).$$

(b) Given $r_1 = 2i + j + k, r_2 = i - 3j + 2k, r_3 = 2i - j - 3k, r_4 = 3i - 2j - 5k$, determine scalars λ_1, λ_2 and λ_3 so that

$$r_4 = \sum_{i=1}^{3} \lambda_i r_i.$$

(c) Let u and v be non-collinear vectors and

$$P = (2x + y + 1)u + (x + 4y)v, \ Q = (2x - 3y - 1)u + (y - 2x + 2)v$$

determine $2x - 3y$ if $3P - 2Q = 0$.

2. (a) A particle moves so that its position vector is given by $r = \cos \omega t i + \sin \omega t j$ where ω is a constant. Show that

(i) the velocity v of the particle is perpendicular to r,

(ii) the acceleration a is directed toward the origin and its magnitude proportional to the distance from the origin,

(iii) $r \times v$ is a constant vector.

(b) (i) If $P = \sin ti + \cos tj + tk$, $Q = \cos ti - \sin tj - 3k$ and $R = 2i - 3j - k$, find

$$\frac{d}{dt}(P \times Q \times R)$$

at $t = 0$.

If e_1 and e_2 are constant vectors and λ is a scalar, and given that $H = e^{-\lambda x}(e_1 \sin \lambda y + e_2 \cos \lambda y)$, determine

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}.$$

3. (a) Let $F = (4xy - 3x^2z^2)i + 2x^2j - 2x^3zk$ and $r = xi + yj + zk$, a space curve. Show that $\int_\Gamma F \cdot dr$ is independent of the curve Γ joining any two given points. Show also that there exists a differentiable function ϕ such that $F = \nabla\phi$.

(b) Consider $A = \dfrac{-yi + xj}{x^2 + y^2}$. Compute

(i) $\nabla \times A$ and

(ii) $\oint A \cdot dr$ around any closed curve.

4. (a) Let P be a scalar field and U a vector field. What do you understand by

(i) gradient of P, (ii) divergence U and (iii) curl of U.

(iv)Show that $\nabla \cdot (PU) = (\nabla P) \cdot U + U(\nabla \cdot P)$.

(b) (i)If $q = c \times r$, where c is a constant vector and $r = xi + yj + zk$, show that $c = \frac{1}{2}(\nabla \times q)$.

(ii) Determine constants $\lambda_1, \lambda_2 and \lambda_3$ such that $u = (x + 2y + \lambda_1 z)i + (\lambda_2 x - 3y - z)j + (4x + \lambda_3 y + 2z)k$ is conservative.

5. (a) State (i) the divergence theorem, (ii) the Green's theorem and, (iii) the Stoke's theorem.

(b) (i) If $F = \nabla\phi$ and $\nabla \cdot F = 0$, show that

$$\int\int\int_R (F \cdot F) d\tau = \int\int_S \phi F.ndS,$$

where S is a closed surface enclosing the region R.

(ii) Use Stoke's theorem to evaluate the line integral

$$\int_{\Gamma} e^{-x} a.dr$$

where $a = \imath \sin y + \jmath \cos y$ and Γ is the rectangle with vertices $(0,0,0), (\pi,0,0), (\pi,\frac{\pi}{2},0), (0,\frac{\pi}{2},0)$.

6. (a) Given that $x = \rho \sin\theta \cos\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho\cos\theta$, determine

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}.$$

Hence compute

$$\int\int\int_V \nabla \cdot A dV,$$

where $A = xz^2\imath + (x^2y - z^3)\jmath + (2xy + y^2z)k$ and V is the region bounded by the hemisphere and the $xy-$plane.

(b) (i) Consider

$$A_i = B_{ik}m_k,$$

$$m_k = C_{kl}n_l; \quad (i,k,l = 1,2).$$

Expand the repeated sum completely.

(ii) Write the terms of $\overline{g}_{rs} = g_{jk}\frac{\partial x^j}{\partial \overline{x}^r}\frac{\partial x^k}{\partial \overline{x}^s}$, $N = 3$.

(iii) If

$$x_1 = \overline{x}_{11}\overline{y}_1 + \overline{x}_{12}\overline{y}_2$$

$$x_2 = \overline{x}_{21}\overline{y}_1 + \overline{x}_{22}\overline{y}_2$$

and

$$\overline{y}_1 = z_{11}u_1 + z_{12}u_2$$

$$\overline{y}_2 = z_{21}u_1 + z_{22}u_2$$

write the expression for the x_i's in index notation.