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MTS 421 (COMPLEX ANALYSIS II): LECTURE GUIDE

1. Revise the MTS 321 notes.

2.0 <u>ZEROS AND POLES.</u>

Suppose $z = z_0$ is an isolated singularity of a complex function f, and that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k \qquad (2.1)$$

is the Laurent series representation of f valid for the punctured open disk

$0 < |z - z_0| < R$.

The part

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} \qquad\qquad (2.2)$$

is called the principal part of the Luarent series (2.1)

NOTE:

(i) If the principal part is zero, that is, all the coefficients a_{-k} in (2.2) are

zero, then $z = z_0$ is called a **removable singularity.**

(ii) If the principal part contains a finite number of nonzero terms, then $z = z_0$ is

called a **pole**. If, in this case, the last nonzero coefficient in (2.2) is a_{-n} , $n \geq 1$, then

we say that $z = z_0$ is a **pole of order n**. If $z = z_0$ is pole of order 1, then the principal

part (2.2) contains exactly one term with coefficient a_{-1}. A pole of order 1 is

commonly called a **simple pole**.

(iii) If the principal part (2.2) contains an infinitely many nonzero terms, then $z = z_0$ is

called an **essential singularity**.

2.1 MEROMORPHIC FUNCTION

A function f is meromorphic if it is analytic throughout a domain D, except possibly for poles in D. It can be proved that a meromorphic function can have at most a finite number of poles in D. For example, the rational function $f(z) = \dfrac{1}{(z^2+1)}$ is meromorphic in the complex plane.

Theorem 2.1:A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written

$$f(z) = (z - z_0)^n \phi(z)$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) = 0$.

Proof:

Theorem 2.2: A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written

$$f(z) = (z - z_0)^{-n} \phi(z)$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) = 0$.

Proof:

Theorem 2.3: If the functions g and h are analytic at $z = z_0$ and h has a zero of order n at $z = z_0$ and $g(z_0) = 0$ then the function $f(z) = \dfrac{g(z)}{h(z)}$ has a pole of order n at $z = z_0$.

Proof:

Theorem 2.4 : (Argument Principle)   Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C . Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} = N_0 - N_P$$

where N_0 is the total number of zeros of f inside C and N_P is the total number of poles of f inside C.   In determining N_0 and N_P, zeros and poles   are counted according to their order or multiplicities.

Proof:

Theorem 2.5 (Rouche's theorem) : Let C be a simple closed contour lying entirely within a domain D. Suppose f and g are analytic in D. If the strict inequality

$|f(z) - g(z)| < |f(z)|$ holds for all z on C, then f and g have the same number of zeros (counted according to their order or multiplicities) inside C.

Proof:

2.1 Summation of infinite series

We shall discuss the *Summation of infinite series* as one of the consequences of the *Residue theorem*. Hence, the students are advised to revise MTS 321 before the class.

3.0 Analytic continuation

Suppose we are given an infinite power series or an analytic formula which defines an analytic function f in a domain D. It is natural to ask whether one can extend its domain of analyticity. More precisely, can we find a function F which is analytic in a

larger domain and whose values agree with those of f for points in D? Here, F is called an analytic continuation of f . For example, the complex exponential function e^z is the analytic continuation of the real exponential function e^x defined over the real interval $(-\infty, \infty)$. The complex function e^z is analytic in the finite complex plane and $e^z = e^x$ when $z = x$, x being real. More generally, suppose f_1 is analytic in a domain D_1 and f_2 is analytic in another domain D_2. If $D_1 \cap D_2 \neq \phi$ and $f_1(z) = f_2(z)$ in the common intersection $D_1 \cap D_2$, then f_2 is said to be the analytic continuation of f_1 to D_2 and f_1 is the analytic continuation of f_2 to D_1.

Example;

- For definition of analytic function, revise MTS321.

4. Conformal Mappings and Applications

Here, we introduce various techniques for effecting the mappings of regions. Two special classes of transformation, ***the bilinear transformations*** and ***the Schwarz–Christoffel transformations,*** will be discussed. A bilinear transformation maps the class of circles and lines to the same class, and it is conformal at every point except at its pole. The Schwarz–Christoffel transformations take half-planes onto polygonal regions. These polygonal regions can be unbounded with one or more of their vertices at infinity.

5. BOUNDARY VALUE PROLEMS

The potential field problems, including potential fluid flows, steady state temperature distribution, electrostatics problems and gravitational potential problems are governed by the Laplace equation. There is no time variable in these problems, and the characterization of individual physical problems is exhibited by the corresponding prescribed boundary conditions. The mathematical problem of finding the solution of a partial differential equation that satisfies the prescribed boundary conditions is

called a boundary value problem, of which there are two main types: Dirichlet problems where the boundary values of the solution function are prescribed, and Neumann problems where the values of the normal derivative of the solution function along the boundary are prescribed. In other physical problems, like the heat conduction and wave propagation models, the time variable is also involved in the model. To describe fully the partial differential equations modelling these problems, one needs to prescribe both the associated boundary conditions and the initial conditions. The latter class is called an initial-boundary value problem. Here, we discuss some of the solution methodologies for solving boundary value problems and initial-boundary value problems using complex variables methods.

The link between analytic functions and harmonic functions is exhibited by the fact that both the real and imaginary parts of a complex function that is analytic inside a domain satisfy the Laplace equation in the same domain. *The Gauss mean value theorem states that the value of a harmonic function at the center of any circle inside the domain of harmonicity equals the average of the values of the function along the boundary of the circle.*

The maximum principle will be used to discuss the solutions of the problems.

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4. Sansonne, J. and Gerretsen, J., Lectures on the Theory of Functions

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