MTS 101 LECTURE 4 : MATRICES AND MATRIX ALGEBRA

DEFINITION

A matrix is a rectabgular array of the elements of a field (i.e. on array of numbers). Thus if m, n are two positive integers ≥ 1 and F is a field ($\mathbb{R}$ *or* $\mathbb{C}$) then the array:

$$= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & --- & - \\ - & - & --- & - \\ a_{m1} & a_{m2} & --- & a_{mn} \end{pmatrix}$$

is called an *m* x *n* matrix in F (since it contains m rows and n columns)

Its first row is $(a_{11} \quad a_{12} \quad \dots \quad a_{1n})$ and first column is $\begin{pmatrix} a_{11} \\ a_{12} \\ - \\ - \\ a_{m1} \end{pmatrix}$

The numbers that constitute the matrix are called its ELEMENTS.

Let a_{ij} denote the element of the matrix in the i^{th} row an j^{th} column. Then for ease of notation we can denote our *m* x *n* matrix by

(a_{ij}) i = 1,2,….,n j=1,2,…..,m or simply by capital letter $A_{m \times n}$

Order

The order of a matrix is the no of rows and columns e.g (a_{ij}) is of order *m* x *n*.

If m=n, then the matrix is called a SQUARE MATRIX of order n.

Definition 2: Row and Column Matrices

A rectangle matrix consisting of only a single row is called a ROW MATRIX e.g. (1,2,3,4). Similarly, a rectangle matrix consisting of a single column is called a COLUMN MATRIX e.g.

$$\begin{pmatrix} 3 \\ 5 \\ 7 \\ 4 \end{pmatrix}$$

Definition 3: Null Matrix

This is a matrix having each of its elements = 0 e.g $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Definition 4: Diagonal Element, Diagonal Matrix

The elements a_{ij} of a matrix (a_{ij}) are called its DIAGONAL ELEMENTS (or elements of the main diagonal)

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **, a_{11} = 1, a_{22} =5, a_{33} = 9**

A square matrix in which all the elements other than the diagonal elements are zero is called a DIAGONAL MATRIX.

$$\text{Viz} = \begin{pmatrix} d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & d_3 & 0 & --- & 0 \\ - & - & --- & - & --- & 0 \\ 0 & 0 & 0 & 0 & --- & d_n \end{pmatrix} \quad \text{e.g.} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Such a matrix is denoted (d_1, d_2, …., d_n) or (d_i, d_{ik}) for I, k = 1,2, …n where d_{ii}=1, d_{ik} = 0 ($i \neq k$)

NB: Its diagonal elements may also be zero

e.g. i. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ii. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ iii. $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

are diagonal matrices.

Definition 5: Scalar and Scalar Matrix

A diagonal matrix where diagonal elements are all equal is called a SCALAR MATRIX.

e.g. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Definition 6: Identity Matrix (or Unit Matrix)

A diagonal matrix whose elements are each equal to 1 is called and IDENTITY MATRIX. It is denoted In (or I_{nxn}) .

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Definition 7: Symmetry

A square matrix where elements are arranged symmetrically about the main diagonal is called a SYMMETRIC MATRIX.

e.g. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

On the other hand, if for a square matrix, there is no symmetric about the main diagonal but for every element a_{ij} on one side of the main diagonal, there is a corresponding – a_{ij} on the other side, then the matrix is a SKEW-SYMMETRIC MATRIX.

Furthermore, the diagonal elements are all zero e.g.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & -\frac{1}{2} & 2 \\ -2 & \frac{1}{2} & 0 & 1 \\ -3 & 2 & -1 & 0 \end{pmatrix} \rightarrow \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

Definition 8: Triangular matrix

A square matrix whose elements a_{ij} are all zero whenever $i < j$ is called a LOWER TRIANGULAR MATRIX.

A square matrix whose elements $a_{ij} = 0$ whenever $i > j$ is called an UPPER TRIANGULAR MATRIX.

Hence, a diagonal matrix is both upper and lower matrix.

e.g. $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 3 & 2 \end{bmatrix}$, $\rightarrow$ Lower Triangular Matrix

$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 3 \end{bmatrix}$, $\rightarrow$ Upper Triangular Matrix

MATRIX ALGEBRA

Equality of Matrices

A and B are equal if

i. they are of the same order

ii. their corresponding elements are the same

Addition of Matrices

If A and B are of the same order, the their sum is a matrix C of the same order whose elements are the sums of the corresponding elements of A and B.

$\Longrightarrow C = A + B$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$C_{ik} = a_{ik} + b_{ik} \qquad i = 1,2, \ldots\ldots\ldots m, \;\; k = 1,2, \ldots\ldots\ldots\ldots n$

* Only matrices of the same order can be added.

Properties of Matrix Addition

i. Matrix addition is commutative $\Longrightarrow$ $A + B = B + A$

ii. Matrix addition is Associative $\Longrightarrow$ $(A + B) + C = A + (B + C)$

iii. If 0 is a null matrix of the same order as A, the $A + 0 = 0 + A = A$

iv. To each A there exists a matrix B of the same order s.t $A + B = 0 = B + A$

(i) $\longrightarrow$ (iv) $\Longrightarrow$ Matrix addition is Abelian

Exercise 1.

Find the sum of these matrices and establish their Commutativity

i. $\begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & -3 \\ 7 & 5 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Exercise 2:

Establish the associativity of the following matrices

i. $\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 5 \\ 7 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 6 \\ 2 & 5 & 4 \end{bmatrix}$

ii. $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$

Exercise 3:

Establish the order of each matrix 1,2 and 4 and find the a_{11}, a_{12}, etc what are the diagonal elements.

Exercise 4:

Which of the following matrices are (i) Triangular Matrices, (ii) Unit Matrices (iii) null matrices and Scalar matrices.

i. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ iii. $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 2 & 0 \end{bmatrix}$ iv. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

v. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ vi. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ vii. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ viii. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

ix. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Exercise 5:

i. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ Find a Matrix B such that A + B = 0

ii. If $A = \begin{bmatrix} 2 & 5 & 8 \\ 3 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -5 & -8 \\ -3 & -4 & -6 \end{bmatrix}$ Find A + B

Multiplication by Scalar

Let $A = (a_{ij})$ $i = 1,2,3,\ldots m$ be a matrix, $j = 1,2,3,\ldots n$

And Let α be a scalar (i.e. any number), then $\alpha A = C = (C_{ij})$

Where $\alpha a_{ij} = C_{ij}$ $i = 1,2, \ldots m$, $j = 1,2,\ldots n$

Example

If $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 4a_1 & 4a_2 & 4a_3 \\ 4b_1 & 4b_2 & 4b_3 \end{bmatrix}$

Properties

If A,B are matrices and α,β are scalars, then

i. $\alpha(A + \beta) = \alpha A + \alpha\beta$

ii. $(\alpha + \beta)A = \alpha A + \beta B$

iii. $(\alpha\beta) A = \alpha(\beta A)$ Examples to be given in class

DIFFERENCE OF TWO MATRICES

If two matrices A and B are of the same order, then the difference A – B = A + (-B) = A + (-1)B

Exercise 6:

i. For Exercice (2) above, find (A – B) + C

ii. A – B – C iii. 2A + 3B iv. A + 2B + $^1/_2$C

Exercise 7:

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ and B =

i. Find a matrix C such that A + C is a diagonal matrix.

ii. Find a matrix D such that A + B = 2D.

iii. Find a Matrix E such that (A + B) + E is zero matrix.

MULTIPLICATION OF MATRICES

The product AB of two matrices exist if the number of columns of A = the number of rows of B.

e.g. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix} = \begin{bmatrix} ap + bx & aq + by & ar + bz \\ cp + dx & cq + dy & cr + dz \end{bmatrix}$

Since A_{2x2} and B_{2x3} i.e, no. of column of A = no. of rows of B, then AB exist.

Let A = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ of order *m* x *n* and B = $\begin{bmatrix} b_{11} & \cdots & b_{iq} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nq} \end{bmatrix}$ of order *n* x *q*

Then AB is the matrix

$$C = \begin{bmatrix} C_{ii} & C_{iq} \\ C_{mi} & C_{mq} \end{bmatrix} \text{ of order } m \times q$$

in which the element C_{ij} is the sum of products (term by term) of elements of i^{th} row of A and the j^{th} column of B. Thus for the matrices A = (a_{ik}), B = (b_{kj}), the product AB is matrix C = (C_{ij})

$$\text{where } C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

* Multiplication is possible if no. of column of the first matrix = no. of rows of the second matrix.

Example:

If A = $\begin{bmatrix} 2 & 5 & 3 \\ 0 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -5 & 0 \\ 0 & 2 & 6 \end{bmatrix}$ Find (i). AB, (ii) BA (iii) A^2 (iv) $5B^2$