1.0 The Straight Line

1.1 The Equation of the Straight Line

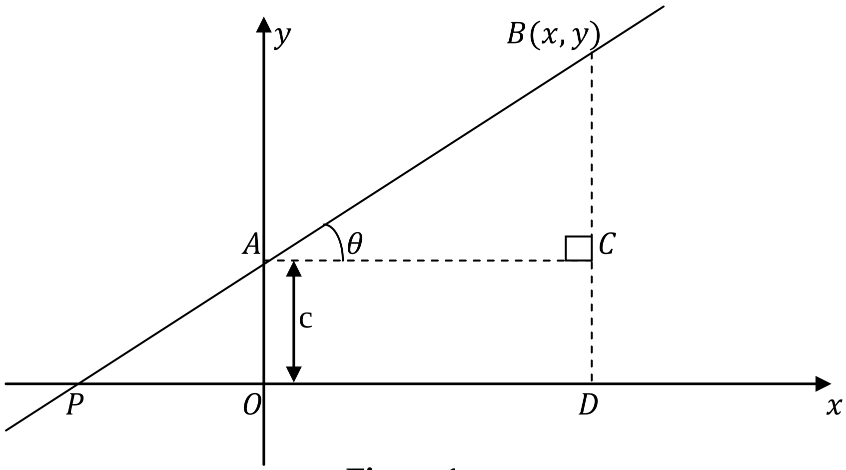


Figure 1

Consider figure 1 in which PAB is the straight line, angle $APO = \theta, OA = c, B(x,y)$ is any point on the straight line, BD is the ordinate and AC is parallel to Ox.

It is clear that

$$< BAC =< APO \qquad \text{(Corresponding angles)}$$

Therefore,

$$\tan\theta = \frac{BC}{AC}$$

$$\tan\theta = \frac{y-c}{x}$$

$$\mathrm{y} = x\tan\theta + c,$$

$\tan\theta$ is the slope of the straight line denoted by m. Thus $y = mx + c$ is the gradient intercept form of the straight line.

1.2 Other Forms of the Equation of a Straight Line

Gradient and One Point Form

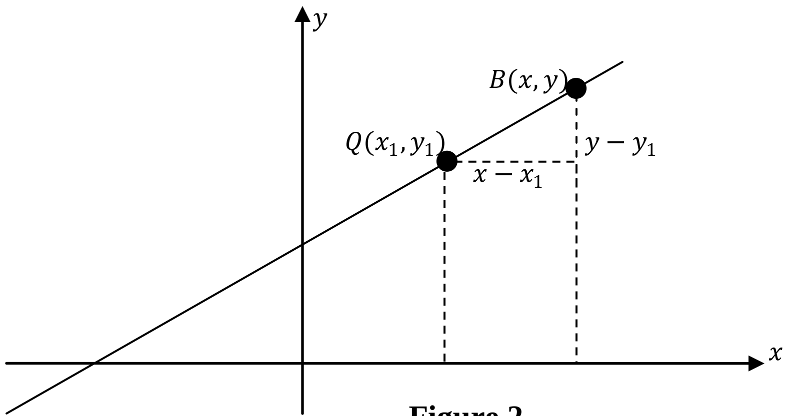


Figure 2

Consider figure 2 in which a straight line is passing through $Q(x, y)$ and having a gradient m. $B(x, y)$ is a variable point on the straight line, thus

$$\text{Gradient of } QB = \frac{y - y_1}{x - x_1}$$

$$m = \frac{y - y_1}{x - x_1}$$

$$\text{Or } y - y_1 = m(x - x_1)$$

Example: Find the equation of a straight line of slope 3, if it passes through the point $(-2, 3)$.

Solution: The equation of a straight line of gradient m passing through (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

$$m = 3, x_1 = -2, y_1 = 3$$

Hence the equation of the straight line is

$$y - 3 = 3(x — 2)$$

$$y - 3 = 3(x + 2)$$

$$y = 3x + 9$$

Two Points Form

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a straight line with slope m. We take a variable point $B(x, y)$ on the line, as shown below.

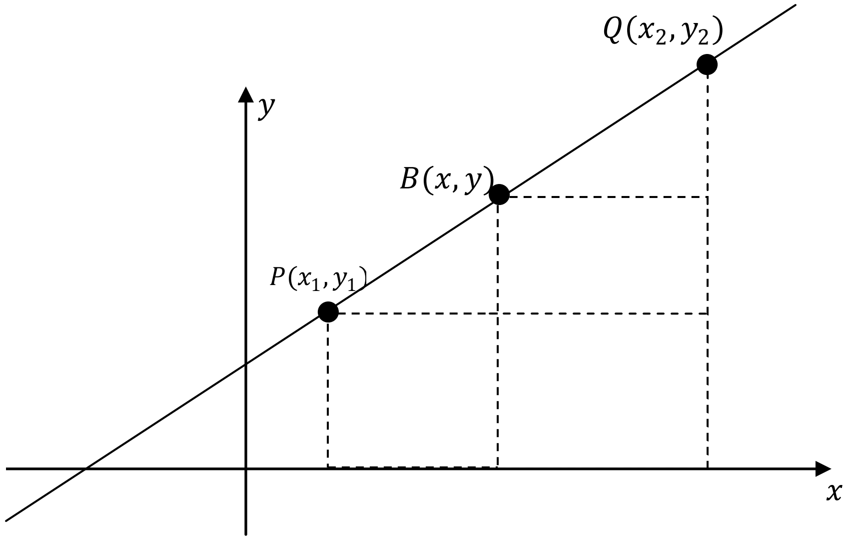


Figure 3

From figure 3,

$$\text{Gradient } PB = \frac{y - y_1}{x - x_1}$$

$$\text{Gradient } PQ = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus the equation of the straight line is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the equation of the straight line which passes through the points $A(2, -3)$ and $B(3,2)$.

Solution

$(x_1, y_1) = (2, -3), (x_2, y_2) = (3, 2)$

Hence the equation of the straight line through A and B is

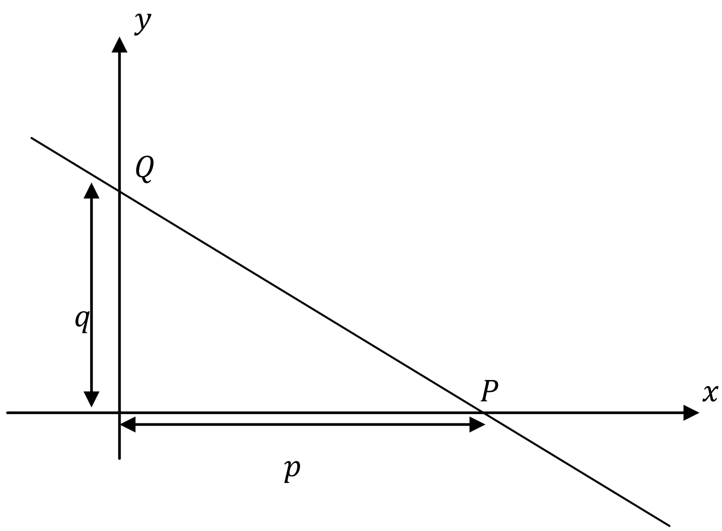
$$\frac{y-(-3)}{x-2} = \frac{2-(-3)}{3-2}$$

$$\frac{y+3}{x-2} = \frac{2+3}{1}$$

$$y + 3 = 5x - 10$$

$$y = 5x - 13$$

The Equation of the Straight Line making Intercepts p and q on Ox and Oy respectively



Let the straight line cross the x and y axes at P and Q respectively. Let the coordinate at P be $(p, 0)$ and at Q be $(0, q)$. Thus, using the two point form formula,

$$\frac{y-q}{x} = \frac{0-q}{p}$$

$$py - pq = -xq$$

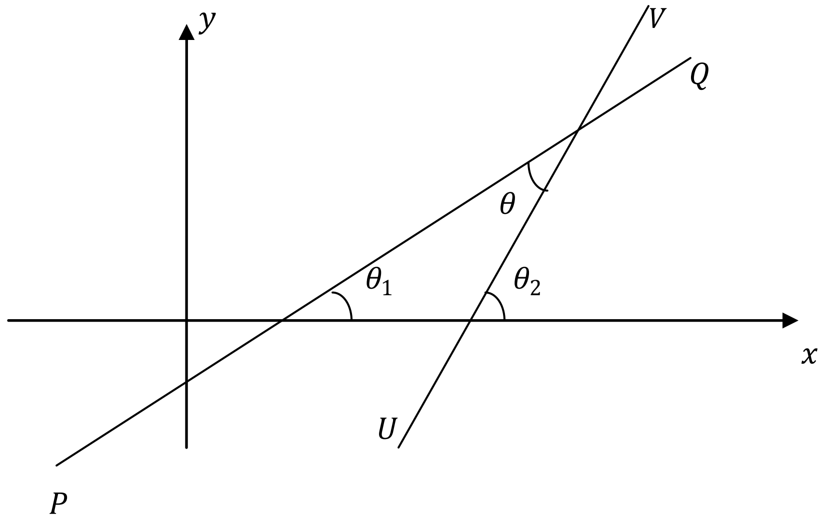
$$py + xq = pq$$

$$\therefore \ \frac{y}{q} + \frac{x}{p} = 1$$

This is called the intercept form of the equation of the straight line.

Other forms of equation of a straight line exist. The reader should endeavour to find out.

1.3 The Angle Between Two Straight Lines



Let PQ and UV be two given straight lines (as shown in the figure above) with gradients m_1 and m_2 respectively. The acute angle between the lines is θ and $\theta = \theta_2 - \theta_1$. From our previous knowledge $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$. Thus

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}\\ &= \frac{m_2 - m_1}{1 + m_1 m_2}\end{aligned}$$

Hence $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

If $\dfrac{m_2 - m_1}{1 + m_1 m_2}$ is negative, we obtain the obtuse angle $180 - \theta$. The acute angle θ between the two lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Remark

1. If $\tan\theta = 0$, the two lines are parallel. Then $m_2 = m_1$.
2. If the lines are perpendicular to each other, then $\theta = 90^o$ and $\tan\theta = \infty$.

Therefore,

$$1 + m_1 m_2 = 0$$

$$m_1 m_2 = -1$$

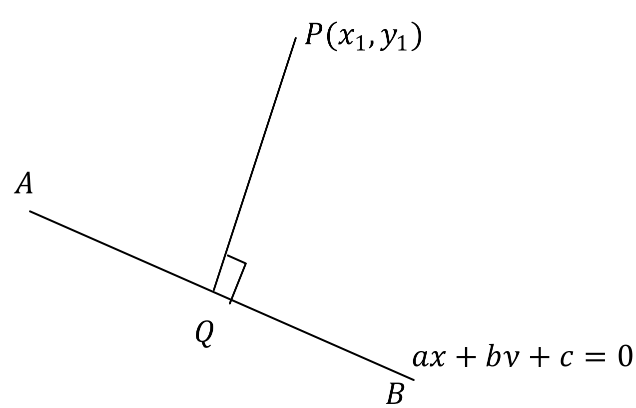
$$m_1 = \frac{-1}{m_2}$$

1.4 The Perpendicular Distance of a Point from a Straight Line

Let $P(x_1, y_1)$ be the point and the equation of the line be

$$ax + by + c = 0 \qquad (1)$$

as shown below.



Let the straight line $ax + by + c = 0$ be denoted AB and the line perpendicular to it be PQ. From the equation $ax + by + c = 0$

$$y = \frac{-a}{b}x - \frac{c}{b}$$

Thus, the gradient of line AB is $\frac{-a}{b}$. Therefore the gradient of the line PQ is $\frac{b}{a}$.

Using the point formula, the equation of the straight line PQ is derived thus,

$$\frac{b}{a} = \frac{y - y_1}{x - x_1}$$

Hence, the equation of the line PQ is

$$-bx + ay = ay_1 - bx_1 \qquad (2)$$

Solving (1) and (2) simultaneously to determine the coordinate of $Q(x_2, y_2)$, we have that

$$x_2 = \frac{b^2x_1 - aby_1 - ac}{a^2 + b^2} \text{ and}$$

$$y_2 = \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2}$$

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2} - x_1\right)^2 + \left(\frac{a^2y_1 - abx_1 - bc}{a^2 + b^2} - y_1\right)^2}$$

$$= \sqrt{\frac{a^2(ax_1 + by_1 + c)^2 + b^2(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}}$$

$$= \sqrt{\frac{(a^2 + b^2)(ax_1 + by_1 + c)^2}{(a^2 + b^2)^2}}$$

$$= \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

is the length of the perpendicular.

Example: Find the length of the perpendicular from the point $P(2, -4)$ to the line $3x + 2y - 5 = 0$.

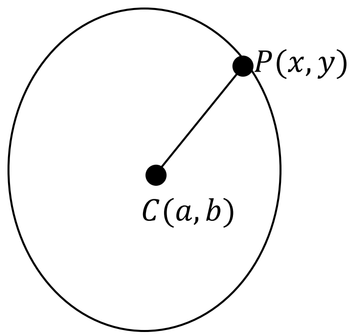
Solution

The length is $\dfrac{3(2) + 2(-4) - 5}{\sqrt{3^2 + 2^2}} = \dfrac{6 - 8 - 5}{\sqrt{13}} = \dfrac{-7}{\sqrt{13}}$

Hence the perpendicular distance is $\frac{7}{\sqrt{13}}$.

2.0 The Equation of a Circle

Consider a circle with centre $C(a, b)$ and radius r (as shown below). Let $P(x, y)$ be an arbitrary point on the circumference of the circle.



It follows that

$$CP = r$$

$$CP^2 = r^2$$

Using the expression for the distance between two points, we have

$$(x - a)^2 + (y - b)^2 = r^2 \qquad (2.1)$$

Or

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2 \qquad (2.2)$$

If we let $a = b = 0$, the centre of the circle is the origin and the equation reduces to

$$x^2 + y^2 = r^2 \qquad (2.3)$$

If we let $a = -g, b = -f$ and $r = \sqrt{g^2 + f^2 - c}$, the equation of a circle is thus of the form

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

This implies that the circle has centre $(-g, -f)$.

Example

Find the equation of the circle with centre $(3, 7)$ radius 5.

Solution

The equation is:

$$(x-3)^2+(y-7)^2=5^2$$

$$x^2-6x+9+y^2-14y+49=25$$

$$x^2+y^2-6x-14y+33=0$$

Exercise

Find the centre and radius of the circle $4x^2+4y^2-12x+5=0$.

2.2 The of a Circle through Three Non-collinear Points

Let the three points $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ be points on the circumference of the circle $x^2+y^2+2gx+2fy+c=0$. Since the circle passes through all the three points, the coordinates of each point must satisfy the equation of the circle.

Hence

$$x_1{}^2+y_1{}^2+2gx_1+2fy_1+c=0$$

$$x_2{}^2+y_2{}^2+2gx_2+2fy_2+c=0$$

$$x_3{}^2+y_3{}^2+2gx_3+2fy_3+c=0$$

Solving these equations simultaneously we obtain the values ofg,f and c.

Example: Find the equation of the circle through the points $(0,0),(3,1)$ and $(5,5)$ and determine the radius.

Solution

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$. Then since $(0,0)$ lies on the circle, $c=0$.

Similarly,

$$9 + 1 + 6g + 2f = 0$$

and

$$25 + 25 + 10g + 10f = 0$$

Solving these simultaneous equations, we have $f = -5, g = 0$.

Hence the required equation is:

$$x^2 + y^2 - 10y = 0$$

2.3 The Equation of the Tangent at the Point (x_1, y_1) on a Circle

Let $P(x_1, y_1)$ be a point on the circumference of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We need to determine the equation of the tangent to the circle at the point (x_1, y_1).

Differentiating the equation with respect to x, we have

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0,$$

Therefore

$$\frac{dy}{dx} = \frac{-(x+g)}{(y+f)}$$

is the gradient of the tangent at the point (x_1, y_1). Thus the equation of the tangent is

$$\frac{-(x+g)}{(y+f)} = \frac{y - y_1}{x - x_1}$$

$$xx_1 + yy_1 + gx + fy = {x_1}^2 + {y_1}^2 + gx_1 + fy_1$$

Adding $gx_1 + fy_1 + c$ to both sides yields

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = {x_1}^2 + {y_1}^2 + 2gx_1 + 2fy_1 + c = 0$$

Since (x_1, y_1) lies on the circle. Hence the required equation is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Exercises

(i) Given the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, determine the length of the tangent from a point $P(X, Y)$ outside the circle.

(ii) Given the circle $x^2 + y^2 = r^2$ and the straight line $y = mx + c$, determine their points of intersection.