Course Title: Real Analysis II

Course Code: MTS 323

Instructor: Dr. Mewomo O.T.

Office Hours: TBA

Sylabus:

1. Differentiation in $\Re^n$
   (a) differentiation of real valued functions of real variable
   (b) directional derivatives
   (c) partial derivatives
   (d) higher order derivatives
   (e) Taylor's theorem
   (f) classification of stationary points
   (g) local extrema with constraints (method of Langrange's multipliers)
2. Integration in $\Re$
   (a) Riemann integral review
   (b) Riemann Stieljes integral
3. Function of bounded variation.
4. Sequences of functions
   (a) pointwise convergence

(b) uniform convergence

(c) Uniform Cauchy criterion

Textbooks: The following are recommended:

1. R.G. Bartle; The elements of real analysis, 2nd edition, Jossey-Bass, (1976).
2. W.R. Wade; An introduction to analysis, 2nd edition, Prentice, (2000).
3. T.M. Apostol; Mathematical analysis.
4. W. Rudin; Mathematical analysis.

Grading:

The grading will be based on weekly homework assignment (10 percent), an in class - mid term test (20 percent) and a final examination (70 percent).

1 Differentiation in $\Re^n$

We recall the definition of derivative of a function f with $Domain(f) \subset \Re$ and $Range(f) \subset \Re$.

Definition 1.1 *Let $x_\circ$ be an interior point of $Dom(f)$. A real number l is called the derivative of f at $x_\circ$, if*

$$lim_{h\to 0}\frac{f(x_\circ + h) - f(x_\circ)}{h} = l,$$

when this exist, we denote it by $f'(x_\circ)$ and so we write

$$lim_{h\to 0}\frac{f(x_\circ + h) - f(x_\circ)}{h} = f'(x_\circ) \quad (h = x - x_\circ) \tag{1.1}$$

We may generalized this notion to a map $f : A \to \Re^m$ where $A \subset \Re^n$.

Definition 1.2 *Let f be a function with $Dom(f) \subset \Re^n$ and $Range(f) \subset \Re^m$. Let a be an interior point of $Dom(f)$ and let u be any vector in $\Re^n$, a vector $v \in \Re^m$ is called the directional derivative of f at a along the line determined by u if*

$$lim_{t\to 0}\frac{f(a+tu)-f(a)}{t} = v.$$

When this limit exit, we denote it by $D_u f(a)$. That is, the directional derivative of f at a in the direction of u is given by

$$D_u f(a) = lim_{t\to 0}\frac{f(a+tu)-f(a)}{t} \tag{1.2}$$

Using ϵ, δ notation, $D_u f(a)$ may be written as follows: Given $\epsilon > 0$, there exists a $\delta, \delta(\epsilon) > 0$, such that $0 < |t| < \delta(\epsilon)$, then

$$|D_u f(a) - \frac{f(a+tu)-f(a)}{t}| < \epsilon$$

Remark: If $a = (a_1, a_2, ..., a_n)$ and $u = (u_1, u_2, ..., u_n)$, then we define $D_u f(a)$ as

$$D_u f(a) = lim_{t\to 0}\frac{f(a_1+tu_1, a_2+tu_2, ..., a_n+tu_n) - f(a_1, a_2, ..., a_n)}{t}$$

Definition 1.3 *Let f be a function with $Dom(f) \subset \Re^n$ and $Range(f) \subset \Re^m$. Let a be an interior point of $Dom(f)$. The directional derivative of f at a in the directions of the special vectors $e_1, e_2, ...e_n$ (basis of $\Re^n$) are called the partial derivatives of f with respect to the 1st, 2nd,...nth variable respectively. Thus if $a = (a_1, a_2, ..., a_n)$, then since $e_1 = (1, 0, 0, ...0)$,*
$e_2 = (0, 1, 0, ...0), ...e_n = (0, 0, ..., 1)$, we have from the above remark that

$$D_{e_1} f(a) = lim_{t\to 0}\frac{f(a_1+t, a_2, ..., a_n) - f(a_1, a_2, ..., a_n)}{t}$$

$$D_{e_2}f(a) = lim_{t\to 0}\frac{f(a_1a_2 + tu_2, ..., a_n) - f(a_1, a_2, ..., a_n)}{t}$$

and in general

$$D_{e_n}f(a) = lim_{t\to 0}\frac{f(a_1, a_2, ..., a_n + t) - f(a_1, a_2, ..., a_n)}{t}$$

The vectors $D_{e_1}f(a), D_{e_2}f(a), ...D_{e_n}f(a)$ are often written as $D_1f(a), D_2f(a), ..., D_nf(a)$.

Remark:

If $m = 1$, then f is a real-valued function, and $D_1f(a), D_2f(a), ..., D_nf(a)$ are usually written as $\frac{\partial f(a)}{\partial x_1}, \frac{\partial f(a)}{\partial x_2}, ...\frac{\partial f(a)}{\partial x_n}$.

Example 1.4 *Let $f : \Re^2 \to \Re$ be defined by $f(x, y) = x^2 + y^2$. Find the directional derivative of f at $a = (2, 1)$ along a line determined by the vector $u = (3, -4)$.*

Solution To be provided in class.

Remark:

The existence of partial derivatives does not imply the existence of directional derivatives as the following example shows.

Example 1.5 *Define $f : \Re^2 \to \Re$ by*

$$f(x, y) = \begin{cases} 7 & if\ x.y = 0 \\ 5 & if\ x.y \neq 0 \end{cases}$$

Assignment 1

1. If $f : \Re^3 \to \Re$ is defined by $f(x, y, z) = 2x^2 - y + 6xy - z^3 + 3z$. Calculate the directional derivative of f at the origin in the direction of the vectors

(a) $u = (1, 2, 0)$

(b) $u = (2, 1, -3)$

Find $D_1 f(0, 0, 0), D_2 f(0, 0, 0)$ and $D_3 f(0, 0, 0)$.

2. Let $f : \Re^2 \longrightarrow \Re$ be defined by

$$f(x, y) = \begin{cases} \frac{2xy^2}{x^2+y^4} & \text{if } (x, y) \neq 0 \\ 0 & \text{if } (x, y) = 0 \end{cases}$$

(a) Show that the partial derivative f_x and f_y exist at the origin and at any point $(w_1, w_2) \neq (0, 0)$.

(b) Show that f is discontinuous at the origin.

Higher Derivatives:

Let $f : \Re^n \longrightarrow \Re^k$ and let a be any point of $\Re^n$ and suppose that one of the partial derivatives $D_j f$ exist for all x in some open ball $S(a, \delta) \subset \Re^n$. Define a new function g on $S(a, \delta)$ by $g(x) = D_j f(x)$, then g is a function with $Dom(g) \subset \Re^n$ and $Range(g) \subset \Re^k$. In addition if $a \in Dom(g)$, then we may ask if $D_k g(a) = D_k D_j f(a)$. If this partial derivative exist, we call it the second partial derivative of f at a with respect to j variable and the k^{th} derivative. Putting x as $(x_1, x_2, ..., x_n)$, then in a more familiar notation, we write

$$D_k D_j f(a) \ \ as \ \ \frac{\partial^2 f(a)}{\partial x_k \partial x_j}$$

Example 1.6 *Let* $f : \Re^2 \longrightarrow \Re$ *be defined by*

$$f(x, y) = xy + \sqrt{1 + x^2} sin \left(\frac{x}{\sqrt{1 + x^2}} \right)$$

Then

$$D_1 f(x,y) = \frac{\partial f(x,y)}{\partial x} = y + \frac{x}{\sqrt{1+x^2}} sin\left(\frac{x}{\sqrt{1+x^2}}\right) + \frac{1}{1+x^2} cos\left(\frac{x}{\sqrt{1+x^2}}\right)$$

From this, it follows that

$$D_2 D_1 f(x,y) = 1$$

Also,

$$D_2 f(x,y) = \frac{\partial f(x,y)}{\partial y} = x$$

$$D_1 D_2 f(x,y) = 1.$$

Thus $D_1 D_2 f(x,y) = D_2 D_1 f(x,y)$.

Definition 1.7 *Suppose* f *has a second partial derivatives on a neighbourhood of* a *which are continuous throughout the neighbourhood. We define* $D^2 f(a)$ *as a function from* $\Re^n \times \Re^n \to \Re$ *by setting*

$$D^2 f(a)(u)^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} D_i D_j f(a) u_i u_j,$$

where $u^2 = (u,u)$ *with* $u = (u_1, u_2, ..., u_n) \in \Re^n$. *The function* $D^2 f(a)$ *is known as the second partial derivative of* f *at* a, *similarly* $D^3 f(a)$ *is the third partial derivative of* f *at* a.

$$D^3 f(a)(u)^3 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D_i D_j D_k f(a) u_i u_j u_k,$$

where $u^3 = (u,u,u)$ *with* $u = (u_1, u_2, ..., u_n) \in \Re^n$.

In general, the higher order derivatives take the following for:
For $u = (h,k)$ and (x,y) in $\Re^2$,

$$D^2 f(a)(u)^2 = h^2 f_{xx}(a) + 2hk f_{xy}(a) + k^2 f_{yy}(a).$$

Once the partial derivatives are continuous, then $f_{xy} = f_{yx}$.

Also,

$$D^3 f(a)(u)^3 = h^3 f_{xxx}(a) + 2h^2 f_{xxy}(a) + 3hk^2 f_{xyy}(a) + k^3 f_{yyy}(a).$$

Taylor's Theorem

For futher details on the notions of continuity, differentiability, Taylor's and Maclaurin's theorems in 1-dimensional see MTS223 lecture note.

We are now in a position to state Taylor's theorem for a real-valued function f with $Dom(f) \subset \Re^n$.

Theorem 1.8 *Let f be a real valued function with $Dom(f) \subset \Re^n$. Let a, b be interior points of $Dom(f)$ and suppose that f has a continuous partial derivatives of order m in an open set containing a, b and the line segement joining a and b. Then there exists c on the line segement such that*

$$f(b) = f(a) + \frac{Df(a)(b-a)}{1!} + \frac{D^2 f(a)(b-a)^2}{2!} + ... + \frac{D^m f(a)(b-a)^m}{m!}$$

If $f(x, y)$ is a function of two variables, then Taylor's theorem takes the form

$$f(x_\circ + h, y_\circ + k) = f(x_\circ, y_\circ) + (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})f(x_\circ, y_\circ) + \frac{(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f(x_\circ, y_\circ)}{2!}$$

$$+ \frac{(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^3 f(x_\circ, y_\circ)}{3!} + ...$$

Note:

With $h = x - x_\circ, k = y - y_\circ$,

$$f(x, y) = \sum_{m=0}^{n-1} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^m f(x_\circ, y_\circ).$$

Example 1.9 *Expand* $f(x, y) = x^2y + 3y - 2$ *in powers of* $x - 1$ *and* $y + 2$.

Solution To be provided in class.

Example 1.10 *Obtain the Maclaurin's series of the function* $f(x, y = e^{x+y} cosy$ *neglecting third and higher degrees.*

Solution To be provided in class.

Extreme Value:

Definition 1.11 *Let* f *be a real value function with* $Dom(f) \subset \Re^n$. *We say that* f *has a local (or relative) maximum (minimum) at a point* $a \in Dom(f)$, *if there is a neighbourhood* $U(a)$ *of* a *such that* $f(x) \leq f(a) \quad (f(x) \geq f(a)$ *for all* $x \in U(a) \cap Dom(f)$.

A function is said to have a local or relative extreme value at a if it either have a relative maximum or relative minimum at a. A point a is called a critical point if $f'(a) = 0$. A critical point that is not a local extreme is called a saddle point.

Theorem 1.12 *Let* f *be a real valued function with* $Dom(f) \subset \Re^n$. *If* f *is differentiable at* a *and* a *is a local extremum, then* $f'(a) = 0$.

Theorem 1.13 *Let* f *be a real valued function with* $Dom(f) \subset \Re^n$. *If* f *is differentiable at* a *and* a *is a local extremum, then*

$$\frac{\partial f(a)}{\partial x_1} = \frac{\partial f(a)}{\partial x_2} = \ldots \frac{\partial f(a)}{\partial x_n} = 0.$$

Corollary 1.14 *If* $f'(a) \neq 0$, *then* f *cannot have an extreme value at* a.

Remark:

We have not said that if $f'(a) = 0$, then f necessarily has an extremum at a.

Example 1.15 *Let* $f : \Re^2 \to \Re$ *be defined by*

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = x^2 - y^2$

Describe the nature of the critical point of f *in each case.*

Solution To be provided in class.

Definition 1.16

1. *A real valued function* g *defined on* $\Re^n$ *is called a quadratic function if it has the form*

$$g(h) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} h_i h_j,$$

$h = (h_1, h_2, ..., h_n) \in \Re^n$ *and some marix* $(a_{ij}), a_{ij} \in \Re, i = 1, 2, ...n,$ $j = 1, 2, ...n.$

2. *A quadratic function* $g : \Re^n \to \Re$ *is said to be*
   (a) *positive definite if* $g(h) \geq 0$ *for every* $h \in \Re^n$ *and* $g(h) = 0$ *only for* $h = 0$
   (b) *negative definite if* $g(h) \leq 0$ *for every* $h \in \Re^n$ *and* $g(h) = 0$ *only for* $h = 0$.

Theorem 1.17 *If* $f : \Re^2 \to \Re$ *and* a *is an interior point of* $Dom(f)$. *Suppose* f *has continuous partial derivatives of order two on a neighbourhood of* a *and that* a *is a critical point of* f.

1. *If $D^2 f(a)(u^2)$ is positive definite, then f has a relative minimum at a.*

2. *If $D^2 f(a)(u^2)$ is negative definite, then f has a relative maximum at a.*

3. *If $D^2 f(a)(u^2)$ assume both positive definite and negative definite for $u \in \Re^n$, then f has a saddle point.*

Example 1.18 *Verify that the origin is a critical point of the function $f(x, y) = x^2 - xy + y^2$ and determine whether at this point f has a local mimimum, local maximum or neither.*

Solution To be provided in class.

Example 1.19 *Discuss the behaviour of the following function defined from $\Re^2$ into $\Re$ defined by*

$$f(x, y) = 2x^5 y + 3xy^5 + xy.$$

Solution To be provided in class.

Remark:

In some situations, the determination of the sign of $D^2 f(a)$ may not be easy, we give the following more practical test.

Theorem 1.20 *Let f be a real-valued function with $Dom(f) \subset \Re^2$, have continuous partial derivate of order three on an open set containing a critical point a in $\Re^2$, and let $\Delta = f_{xx}(a) f_{yy}(a) - (f_{xy}(a))^2$.*

1. *If $\Delta > 0$ and if $f_{xx}(a) > 0$, then f has a relative minimum at a.*

2. *If $\Delta > 0$ and if $f_{xx}(a) < 0$, then f has a relative maximum at a.*

3. If $\Delta < 0$, then the point a is a saddle point of f.

We next consider extreme problems with some copnstraints which can be put in the form maximise or minimise $P = f(x, y)$ subject to the condition $g(x, y) = 0$.

Example 1.21 *Suppose a manufacturer is producing cements at two locations A and B. Let us assume that the cost of paying for the inspection of the work at both locations depends on the number of inspections x at A and y at B according to the formula $C(x, y) = 2x^2 + xy + y^2 + 25$. How many inspections should be made at each site to minimize his cost if the total number of inspection must be 16?*

Solution To be provided in class.

Example 1.22 *Of all the rectangles having the same perimeter 10 meters. Find the one having the greatest area.*

Solution To be provided in class.

Remark:

In some cases, it is difficult to solve the resulting equation in a closed form for some of the unknown in terms of the other. Thus, we result to the method of Lagrange's multiplier.

Lagrange's Multiplier

In finding extreme values of a function f with $Dom(f) \subset \Re^n$, and $Range(f) \subset \Re$, subject to the restraining condition $g(x) = 0$, we apply the following procedure:

1. Define the function F on $Dom(f) \cap Dom(g)$ into $\Re$ by
   $F(x) = f(x) + \lambda g(x)$.

2. Solve the $n+1$ equations for $\lambda_1, \lambda_2, ...\lambda_n$.
   $F_{x_1} = 0, F_{x_2} = 0, ..., F_{x_n} = 0, g(x) = 0$. The constant λ which is not wanted in the end is called the Lagrange's multipliers.

If there are two restraining conditions $g(x) = 0$ and $h(x) = 0$, we shall consider the function

$$F(x) = f(x) + \lambda g(x) + \beta h(x)$$

and solve the equations $F_{x_1} = 0, F_{x_2} = 0, ..., F_{x_n} = 0, g(x) = 0, h(x) = 0$.

We shall illustrate this method with some examples.

Example 1.23 *Find the shortest distance from the point (1,0) to the parabola* $y^2 = 4x$.

Solution To be provided in class.

Example 1.24 *Find the point on the surface* $x^2 + y^2 + z^2 = 1$ *at which* $f(x, y, z) = xyz$ *is stationary.*

Solution To be provided in class.

Example 1.25 *Find the points on the sphere* $x^2 + y^2 + z^2 = 36$ *that are closest and farthest from the point (1,2,2).*

Solution To be provided in class

Assignment 3:

1. Obtain the Taylor's series of $tan^{-1}(\frac{y}{x})$ about the point (1,1) up to the second degree.

2. Determine the nature of the turning points of the following

   (a) $f(x,y) = x^3y + 3x + y$

   (b) $f(x,y) = sinx + siny + sin(x+y)$ for $0leqx \leq 2\pi, 0 \leq y \leq 2\pi$.

   (c) $f(x,y) = 16 + 4x + 7y - 2x^2 - y^2$

3. Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 - 225 = 0$.

4. Find the point on the plane $2x - 3y - 4z = 25$ which is nearest to the point (3,2,1).

5. Find the maximum and minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25}$ and $z = x + y$.

6. At what point on the curve $x^2 + y^2 = 1$ does the product xy have a maximum?

2 Integration in $\Re$

Riemann Integral and its basic properties were studied in MTS223, for quick review and details on Riemann Integral, see MTS 223 lecture note.

In this section, we shall study Riemann-Stieljes Integral (R-S integral) which is a generalization of Riemann integral.

We begining with the following concepts:

A partition of the closed and bounded interval [a,b] is a finite set of points $P = \{x_0, x_1, ..., x_n\}$, such that $a = x_0 < x_1 < x_2 < ... < x_n = b$. We denote the set of all partitions of [a,b] by $\mathcal{P}_{[a,b]}$.

For any partition P of [a,b], we let

$$M_i = sup\{f(x) : x_{i-1} \leq x \leq x_i\}$$

and

$$m_i = inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$

Definition 2.1 *Let α be a monotone increasing function on [a,b]. Corresponding to each $p \in \mathcal{P}_{[a,b]}$, we write*

$$0 \leq \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \quad i = 1, 2, ..., n.$$

For any bounded real valued function f on [a,b], we put

$$U(p, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i,$$

$$L(p, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

Also, we write

$$\int_a^{\bar{b}} f(x) d\alpha(x) = inf\{U(p, f, \alpha) : p \in \mathcal{P}_{[a,b]}\} \qquad (2.1).$$

This is called the upper R-S integral of f with respect to alpha.

$$\int_{\bar{a}}^{b} f(x) d\alpha(x) = sup\{L(p, f, \alpha) : p \in \mathcal{P}_{[a,b]}\} \qquad (2.2).$$

This is called the lower R-S integral of f with respect to alpha.

If the LHS of (2.1) and LHS of (2.2) are equal, we denote their common value by

$$\int_a^b f d\alpha \quad or \quad \int_a^b f(x)d\alpha(x) \qquad (2.3).$$

If (2.3) exit, we say that f is Riemann-Stieljes (R-S) integrable with respect to α on [a,b].

We recall that a partition Q is said to finer than a partition P (or is called a refinement of P) on the interval [a,b] if P and Q are partitions of [a,b] and $P \subset Q$.

Theorem 2.2 *If p^* is a refinement of p, then*

1. $L(p, f, \alpha) \leq L(p^*, f, \alpha)$
2. $U(p, f, \alpha) \geq U(p^*, f, \alpha)$

Proof To be provided in class.

The next result gives a necessary and sufficient condition for f to be R-S integrable with respect to α.

Theorem 2.3 *If f is a real valued function on [a,b], then f is R-S integrable if and only if given epsilon > 0, there exists a partition $p \in \mathcal{P}_{[a,b]}$ such that*

$$U(p, f, \alpha) - L(p, f, \alpha) < \epsilon \qquad (2.4)$$

Proof To be provided in class.

Theorem 2.4 *If f is continuous on [a,b], then f is R-S integrable on [a,b].*

Proof To be provided in class.

Theorem 2.5 *If f is monotone on [a,b], and α is continuous on [a,b], then f is R-S integrable on [a,b].*

Proof To be provided in class.

Theorem 2.6 *(Basic properties of R-S integral) Let f_1, f_2 be R-S integrable and λ an arbitrary constant. Then*

1. *$f_1 + f_2$ and λf_1 are R-S integrable on [a,b]. Furthermore*

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

   and

$$\int_a^b (\lambda f_1) d\alpha = \lambda \int_a^b f_1 d\alpha$$

2. *If $f_1(x) \le f_2(x)$ for every $x \in [a, b]$. Then*

$$\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$$

3. *If f is R-S integrable on [a,b] and $a \le c \le b$. Then f is R-S integrable on [a,c] and on [c,b]. Furthermore,*

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

3 Function of Bounded Variation

Definition 3.1 *Let f be a function defined on [a,b]. For each $p \in \mathcal{P}_{[a,b]}$, let $W_p(f)$ be the real number given by*

$$W_p(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

If for each $p \in \mathcal{P}_{[a,b]}$, there is a constant $M > 0$, such that $W_p(f) \leq M$ for every $p \in \mathcal{P}_{[a,b]}$, then f is said to be of bounded variation on [a,b].

Since $W_p(f)$ is always less than or equal to M, $p \in \mathcal{P}_{[a,b]}$, then

$$V = \{W_p(f) : p \in \mathcal{P}_{[a,b]}\}$$

is bounded above and non-empty. By completeness axiom, V has a supremum since it is bounded above.

Definition 3.2 *Let f be of bounded variation on [a,b] and for each $p \in \mathcal{P}_{[a,b]}$, the number*

$$v_f(a,b) = sup\{W_p(f) : p \in \mathcal{P}_{[a,b]}\}$$

is called the total variation of f on [a,b].

Remark:

1. $0 \leq v_f(a,b) < +\infty$.
2. If f is of bounded variation on [a,b], then f is bounded on [a,].
3. $v_f(a,b) = 0$ if and only if f is constant.

Theorem 3.3 *If f is of bounded variation on [a,b] and $a < c < b$, then f is of bounded variation on [a,c] and [c,b].*

Proof To be provided in class.

Theorem 3.4 *If f is of bounded variation on [a,b] and $a < c < b$, then*

$$v_f(a,b) = v_f(a,c) + v_f(c,b).$$

Proof To be provided in class.

Theorem 3.5 *If f' exist and is bounded on [a,b], then f is of bounded variation on [a,b].*

Proof To be provided in class.

Remark:

Not every continuous function is of bounded variation. Not that in Theorem 3.5 f is continuous. The following example shows that a continuous function does not need to be of bounded variation.

Example 3.6 *Consider the function*

$$f(x) = \begin{cases} x sin\frac{\pi}{x} & \textit{if}\ 0 < x \leq 2 \\ 0 & \textit{if}\ x = 0 \end{cases}$$

f is clearly continuous on [0,2], we show that f is not of bounded variation on [0,2].

Solution To be provided in class.

We next discuss some basic properties of functions of bounded variation.

Theorem 3.7 *If f and g are of bounded variation on [a,b], then $f + g$ and fg are of bounded variation on [a,b].*

Proof To be provided in class.

Remark:

$f - g$ is of bounded variation on [a,b] if f and g are of bounded variation on [a,b].

Definition 3.8 *Let $f : [a,b] \to \Re$ be of bounded variation on [a,b]. Define $v_f(x) = v_f(a,x), \quad (a \leq x \leq b)$. We call v_f the total variation of f on [a,b].*

Remark:

v_f is monotonic increasing on [a,b] with $v_f(a) = 0$.

Theorem 3.9 *Let $f : [a,b] \to \Re$ be of bounded variation on [a,b]. Then*

1. $v_f(a,y) = v_f(a,x) + v_f(x,y) \quad (a \leq x \leq y \leq b)$.

2. *If f is continuous on [a,b], so is v_f.*

Proof To be provided in class.

Theorem 3.10 *Let f be of bounded variation on [a,b] and let v be defined on [a,b] as follows $v(x) = v_f(a,x)$ if $a < x \leq b, \quad v(a) = 0$. Then*

1. *v is an increasing function on [a,b]*

2. *$v - f$ is an increasing function on [a,b]*

Proof Exercise.

Theorem 3.11 *Let f be of bounded variation on [a,b].Then f can be expressed as the difference of two increasing functions.*

Proof To be provided in class.

Theorem 3.12 *Every monotone increasing function on [a,b] is of bounded variation on [a,b].*

Proof Exercise.

Assignment 4:

1. Show that if f is of bounded variation on [a,b] and $\epsilon > 0$, then there exists $p \in \mathcal{P}_{[a,n]}$ such that $W_p(f) > v_f(a,b) - \epsilon$.

2. Is the function

$$f(x) = \begin{cases} xcos\frac{\pi}{2}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

of bounded variation on [0,1]?

4 Sequences of Functions:Pointwise and Uniform Convergence

In this section, we shall discuss two kinds of convergence of sequence of real-valued functions defined on a subset D of $\Re$- pointwise and uniform convergence. Pointwise convergence is the natural extension of the convergence of sequences and series of numbers, but it lacks many of important desirable properties. The stronger notion of uniform convergence will be shown to posses these properties. We begin with the following definition.

Definition 4.1 *Let (f_n) be a sequnce of real-valued functions defined on a subset D of $\Re$. Then (f_n) is said to converge pointwise on D if for $x \in D$, the sequence of number $(f_n(x))$ converges. If (f_n) converges pointwise on D, then we define $f : D \to \Re$ by*

$$f(x) = lim_{n\to\infty} f_n(x) \qquad for \quad every \quad x \in D.$$

The main interest in sequences of real-valued functions is the following: If (f_n) converges to f, we would like the limit function f to enjoy some of the properties of the individual functions, f_n. For example, let (f_n) converges pointwise to f. The following questions are of interest:

1. If f_n is continuous for each n, is f necessarily continuous?
2. If f_n is integrable for each n, is f necessarily integrable?
3. If f_n is differentiable for each n, is f necessarily differentiable?
4. If f_n is differentiable for each n, and f is also differentiable, does (f_n') converges to f'?
5. If f_n is integrable for each n, is it always true that

$$lim_{n\to\infty}\int_D f_n(x)dx = \int_D lim_{n\to\infty} f_n(x)dx = \int_D f dx.$$

We give examples (in the class) to show that the answer to any of the above five questions is no, if (f_n) converges only pointwise to f. There is another mode of convergence under which all the above questions have affirmative answer under some additional condition. This mode of convergence is called uniform convergence.

Definition 4.2 *Let (f_n) be a sequence of functions defined on subset D of $\Re$. Then (f_n) is said to converge uniformly on D to f If for every $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for every $x \in D$, and for every $n \geq N$.*

We recall that (f_n) converges to f on D pointwise if for each $x \in D$, given $\epsilon > 0$, there exists $N = N(x, \epsilon)$ such that $|f_n(x) - f(x)| < \epsilon$ for every $n \geq N(x, \epsilon)$.

Remark:

The main difference between pointwise and uniform convergence is that for pointwise convergence, the $N \in \mathbb{N}$ depends on ϵ and x, but for uniform convergence, it is possible to find one $N \in \mathbb{N}$ depending only on ϵ that will work for all $x \in D$.

Before given more examples, we give a useful criterion for testing if a given sequence of functions converges uniformly. We begin with the following definition and theorem.

Definition 4.3 *[?] A sequence (f_n) of functions is called uniformly cauchy on a set D if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for every $x \in D$, and for every $n, m \geq N$.*

Theorem 4.4 *(Uniformly Cauchy Criterion) Let (f_n) be a sequence of functions defined on a subset D of $\Re$. Then, there exists a function f such that (f_n) converges uniformly to f on D if and only if the following condition is satisfied: for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for every $x \in D$, and for every $n, m \geq N$.*

Proof To be provided in class.

Theorem 4.5 *Suppose (f_n) is a sequence of functions defined on a subset D of $\Re$. Then (f_n) converges uniformly to f on D if and only if $lim_{n\to\infty}\beta_n = 0$, where $\beta_n = sup_{x\in D}|f_n(x) - f(x)|$.*

Proof Exercise.

We now give more examples.

Example 4.6 *For each* $n \in \mathbb{N}$, *define* $f_n : [0,1] \to \Re$ *by*

$$f_n(x) = \frac{x}{3+nx} \quad for \quad every \quad x \in [0,1], n \geq 1.$$

Show that (f_n) *converges uniformly on [0,1].*

Solution To be provided in class.

Example 4.7 *Let* $f_n(x) = \frac{x^n}{2+x}$ *for* $x \in [0,4]$.

1. *Find the set* $D \subset [0,4]$ *for which* $f(x) = lim_{n\to\infty} f_n(x)$ *is defined as a real-valued function.*

2. *Show that if* $0 < a < 1$, *the convergence is uniform on [0,a].*

3. *Show that the convergence is not uniform on [0,1].*

To be provided in class.

Solution To be provided in class.

Example 4.8 *Let* $f_n(x) = \frac{x^2}{x^2+n}$ *for* $x \in [0,\infty)$. *Show that*

1. $f(x) = lim_{n\to\infty} f_n(x) = 0$, *for all* $x \in [0,\infty)$

2. *the convergence is not uniform on*$[0,\infty)$

3. *the convergence is uniform on [0,a)* $a \in \Re$.

Solution To be provided in class.

We finally give some consequences of uniform convergence of sequences of real-valued functions.

Theorem 4.9 *Let* (f_n) *be a sequence of integrable functions on [a,b]. Suppose* $f_n \to f$ *uniformly on [a,b]. Then*

1. f *is integrable on [a,b]*
2. $lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$

Proof Exercise.

Theorem 4.10 *Let* (f_n) *be a sequence of continuous real-valued functions on [a,b]. and let* f *be a real-valued continuous function on [a,b]. Suppose* (f_n) *is monotone increasing to* f, *as* $n \to \infty$, *for each* $x \in [a,b]$. *Then* $f_n \to f$ *uniformly on [a,b].*

Proof Exercise.

Assignment 5

1. For each $n \in \mathbb{N}$, let

$$f_n(x) = \frac{n^2 x}{1 + n^2 x} \quad (x \in [0,1].$$

   (a) Show that (f_n) converges pointwise on [0,1].

   (b) Is the convergence uniform? Justify your claim.

2. For each integer $n \geq 1$, let

$$f_n(x) = \frac{5}{5 + x^n} \quad (x \in [0,1].$$

   (a) Show that (f_n) converges pointwise on [0,1].

   (b) Is the convergence uniform? Justify your claim.