UNIVERSITY OF AGRICULTURE, ABEOKUTA,DEPARTMENT OF MATHEMATICS

MTS 105-20011/2012 First Semester Lecture note;

COURSE TITLE: Algebra

TOPICS: Binomial Theorem,Binomial Series,Binomial Expansion and Applications

A binomial expression is one that contains two terms connected by a plus or minus sign.Thus $(p+q)$, $(a+x)^2$, $(2x+y)^3$ are examples of binomial expression.

Note:

In order to solve $(a+x)^n$:

1. a decreases in power moving from left to right
2. x increases in power moving from left to right
3. The coefficients of each term of the expansions are symmetrical about the middle coefficient when n is even and symmetrical about the two middle coefficients when n is odd.
4. The coefficients are shown separately below and this arrangement is known as as pascal triangle triangle.A coefficient of a term may be obtained by adding the two adjacent coefficient immediately above in the previous row. This is shown by the triangle below,where for example, $1+3=4, 10+5=15$, and so on.

Pascal triangle method is used for expansion of the form $(a+x)^n$ for integer values of n less than about 8.

n=0:	1							
n=1:	1	1						
n=2:	1	2	1					
n=3:	1	3	3	1				
n=4:	1	4	6	4	1			
n=5:	1	5	10	10	5	1		
n=6:	1	6	15	20	15	6	1	
n=7:	1	7	21	35	35	21	7	1

The numbers in the n-th row represent the binomial coefficients in the expansion of $(a+x)^n$.

Example:

Use the pascal's triangle method to determine the expansion of $(a+x)^7$.

Solution

$$(a+x)^7 = a^7 + 7a^6x + 12a^5x^2 + 35a^4x^3 + 35a^3a^4 + 21a^2x^5 + 7ax^6 + x^7$$

Example:

Determine,using pascal's triangle method,the expansion of $(2p-3q)^5$.

Solution

$$(2p-3q)^5 = (2p)^5+5(2p)^4(-3q)+10(2p)^3(-3q)^2+10(2p)^2(-3q)^3+5(2p)(-3q)^4+(-3q)^5$$

$$= 32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5$$

The binomial Series

The binomial series or binomial theorem is a formula for raising binomial expression ton any power without lengthy multiplication.The general binomial expansion of $(a+x)^n$ is given by

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + ... + x^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r}x^r, \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where for example,3! denotes $3 \times 2 \times 1$ and is termed factorial 3.

With the binomial theorem,n may be a fraction,a decimal fraction, a positive or a negative integer.

Note:

1. ${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!(n-r)!}$ is called binomial coefficient.

2. Since ${}^nC_r ={}^n C_{n-r}$,it follows that the coefficients in the binomial expansion are symmetrical about the middle. There is one middle term (i.e the $\frac{n}{2}$-th term) if n is even,and two middle terms (i.e the $\frac{n-1}{2}$-th and $\frac{n+1}{2}$-th term). If n is odd.

3. The term ${}^nC_r a^{n-r}x^r$ is the $(r+1)$-th term.

In the general expansion of $(a+x)^n$,it is noted that the 4th term is

$$\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3$$

The rth term of the expansion is $\frac{n(n-1)(n-2)...n-(r-2)}{(r-1)!}$

If $a=1$ in the binomial expansion of $(a+x)^n$ then

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$$

which is valid for $-1 < x < 1$.

Example:

Use the binomial series to determine the expansion of $(2+x)^7$

Solution:

The binomial expansion is given by

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3+...$$

when $a=2$ and $n=7$

$$(2+x)^7 = 2^7+7(2)^6x+\frac{(7)(6)}{(2)(1)}2^5x^2+\frac{(7)(6)(5)}{3!}2^4x^3+\frac{(7)(6)(5)(4)}{4!}2^3x^4+\frac{(7)(6)(5)(4)(3)}{5!}2^2x^5+\frac{(7)(6)(5)(4)}{6!}$$

$$= 128+448x+672x^2+560x^3+280x^4+84x^5+14x^6+x^7$$

Example:

Expand $\frac{1}{(1+2x)^3}$ in ascending powers of x as far as the term in x^3,using the binomial series.

Solution

Using the binomial expansion of $(1+x)^n$,where $n=-3$ and x is replaced by $2x$ gives:

$$\frac{1}{(1+2x)^3} = (1+2x)^{-3}$$

$$= 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(2x)^3 + ...$$

$$= 1 - 6x + 24x^2 - 80x^3 + ...$$

The expansion is valid provided $|2x| < 1$

i.e $|x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$

Example:

Expand $\frac{1}{\sqrt{1-2t}}$ in ascending power of t as far as the term in t^3.State the limit of t for which the expression is valid.

Solution

$$\frac{1}{\sqrt{1-2t}} = (1-2t)^{-\frac{1}{2}}$$

$$= 1 + (-\frac{1}{2})(-2t) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-2t)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-2t)^3 + ...$$

Using the expansion for $(1+x)^n$

$$= 1 + t + \frac{3}{2}t^2 + \frac{5}{2}t^3 + ...$$

The expansion is valid when $|2t| < 1$, i.e $|t| < \frac{1}{2}$ or $-\frac{1}{2} < t < \frac{1}{2}$

Example:

Express $\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}}$ as a power series as far as the term in x^2.State the range of values of x for which the series is convergent.

Solution:

$$\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$(1=2x)^{\frac{1}{2}} = 1+(\frac{1}{2})2x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(2x)^2 + ...$$

$$= 1 + x - \frac{x^2}{2} + ...$$

which is valid for $|2x| < 1$ i.e $|x| < \frac{1}{2}$.

$$(1-3x)^{-\frac{1}{3}} = 1 + (-\frac{1}{2})(-3x) + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}(-3x)^2 + ...$$

$$= 1 + x + 2x^2 + ...$$

which is valid for $|3x| < 1$, i.e $|x| < \frac{1}{3}$

Hence

$$\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$= (1 + x - \frac{x^2}{2} + ...)(1 + x + 2x^2 + ...)$$

$$= 1 + x + 2x^2 + xx^2 - \frac{x^2}{2} + ...$$

neglecting terms of higher power than 2

$$1 + 2x + \frac{5}{2}x^2$$

The series is convergent if $-\frac{1}{3} < x < \frac{1}{3}$

Note:

1. Binomial theorem when n is a positive integer

   If a, b are real numbers and n is a positive integer,then

   $$(a+b)^n = a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+...+b^n$$

   or more concisely in terms of the binomial coefficient

   $$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

   we have

   $$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

   where

   $$\binom{n}{0} = \binom{n}{n} = 1$$

2. General form of the binomial theorem when α is arbitrary real number

   If a and b are real numbers such that $|b/a| < 1$ and α is an arbitrary real number,then

   $$(a+b)^\alpha = a^\alpha(1+b/a)^\alpha = a^\alpha(1+\frac{\alpha}{1!}(\frac{b}{a})+\frac{\alpha(\alpha-1)}{2!}(\frac{b}{a})^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}(\frac{b}{a})^3+...)$$

   The series on the right only terminates after a finite number of terms if α is a positive integer in which case the result reduces to the one just given.If α is a negative integer,or a non integral real number,the expression on the right becomes an infinite series that diverges if $1 < |a| > 1$

Example

Expand $(3+x)^{-\frac{1}{2}}$ by the binomial theorem,stating for what values of x the series converges.

Solution

Setting $\frac{b}{a} = \frac{1}{3}x$ in the general form of the binomial theorem gives:

$$(3+x)^{-\frac{1}{2}} = 3^{-\frac{1}{2}}(1+\frac{1}{3}x)^{-\frac{1}{2}} = \frac{1}{\sqrt{3}}(1-\frac{1}{6}x+\frac{1}{24}x^2-\frac{5}{432}x^3+...)$$

The series only converges if $|\frac{1}{3}x| < 1$ and so it is converges provided $|x| < 3$.