UNIVERSITY OF AGRICULTURE, ABEOKUTA,DEPARTMENT OF MATHEMATICS

MTS 101-20011/2012 First Semester Lecture note;

COURSE TITLE:Algebra

TOPIC:COMPLEX ANALYSIS

Complex Numbers

In order to solve equations such as

$$x^2 + 1 = 0$$

or

$$x^2 + 2x + 8 = 0$$

which have no root within the system of real numbers,the number system was extended further to the larger system of complex numbers.

By definition, a complex number is any number x that can be expressed in the form $x = a + ib$ where a and b are real and $i^2 = -1$.The symbol $\mathcal{C}$ is used to denote the system of complex numbers. a is referred to as the real part and b the imaginary part of $a + ib$. Note that the complex numbers include all real numbers. The real numbers can be regarded as complex numbers for which b is zero.

In $\mathcal{C}$, the solution of the equation

$$x^2 + 1 = 0$$

is then $x = \pm\sqrt{-1}$ i.e $x = \pm i$

Algebra of complex Numbers

Let $x = a + ib$ and $y = c + id$ be two complex numbers:

Equality of complex numbers: x and y are equal if their real and imaginary parts are equal i.e $a = c$ and $b = d$

Addition and subtraction of two complex numbers:

The sum of x and y is defined as a complex number $z = x+y = a+ib+c+id = a + c + i(b + d)$

Also,

$w = x - y = a + ib - (c + id) = a - c + i(b - d)$

Multiplication:

$x \times y = (a + ib) \times (c + id) = ac + i^2bd + ibc + iad$

$= ac - db + i(bc + ad)$

Division:

$\frac{x}{y} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c+id)(c-id)}$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Conjugate complex number:

$\overline{x} = a - ib$ is called the conjugate of x.

we have

$$x + \overline{x} = 2a$$

$$x - \overline{x} = 2ib$$

$$x\overline{x} = a^2 + b^2$$

Example: Express in the form $a + ib$

1. $(2 + 4i) + (5 - 2i) = 7 + 2i$
2. $(1 - 8i) - (7 + 2i) = (1 - 7) + (-8 - 2)i = -6 - 10i$
3. $\frac{2+3i}{3+2i} = \frac{2+3i}{3+2i} \times \frac{3-2i}{3-2i}$

$$= \frac{6+6}{9+4} + \frac{9-4}{9+4}i$$
$$\frac{12}{13} + \frac{5}{13}i$$

4. $(1 + 3i)^{-1} = \frac{1}{1+3i} = \frac{1}{1+3i} \times \frac{1-3i}{1-3i} = \frac{1}{10} - \frac{3}{10}i$
5. $(\frac{5(1+i)}{1+3i})^2 = (\frac{5+5i}{1+3i})(\frac{5+5i}{1+3i}) = 3 - 4i$
6. $\frac{2+3i}{i(4-5i} + \frac{2}{i} = \frac{2i-3+2(4i+5)}{-4+5i}$
   $= \frac{22}{41} - \frac{75}{41}i$

Note:

$$i^4 = i \times i^2 = -i$$

$$i^4 = 1$$

$$i^5 = i, i^6 = -1, i^7 = -i$$

and so on.

Example:

Find the solutions of the equation $4x^2 + 5x + 2 = 0$ in the form $\alpha + i\beta$.

Solution:

$x = \frac{-5 \pm \sqrt{-7}}{8}$

$$= -\frac{5}{8} + i\frac{\sqrt{7}}{8}$$

or

$$-\frac{5}{8} - i\frac{\sqrt{7}}{8}$$

Example:

Factorize $a^2 + 3b^2$ as a product of two complex numbers.

Solution:

$a^2 + 3b^2 = a^2 + (b\sqrt{3})^2$

$= (a + ib\sqrt{3})(a - ib\sqrt{3})$

The Argand Diagram

A complex number of the form $z = x + iy$ is specified by the two real numbers x and y.The complex number z may then be made to correspond to a point P with ordered pair of values (x, y) as the co-ordinates of the point P on the plane.

r is known as modulus of the complex number z and is written as $|z|$ or $modz$

$r = |z| = |x + iy| = \sqrt{x^2 + y^2}$

$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

The diagram which represents complex numbers is known as Argand diagram or Argand plane or complex plane.

The angle α between the line OP from the origin to the number and the x-axis is called the argument or amplitudes of the number z.

From the diagram,

$$x = rcos\alpha, y = rsin\alpha$$

$$x^2 + y^2 = r^2, \frac{y}{x} = tan\alpha$$

$$\alpha = argz = tan^{-1}\frac{y}{x}$$

Since on the circle, $\alpha + 2\Pi$ for any integer n,represent the same angle, it follows that the argument of a complex number is not unique such that $-\Pi < Arg(z) \leq \Pi$.

The complex number z can therefore be written as $z = x + iy = rcos\alpha + irsin\alpha = r(cos\alpha + isin\alpha), -\Pi < \alpha < \Pi$.

which is called the modulus-argument form or polar form or trigom=nometric form of the complex number z.

Let $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $Z_2 = r_2(cos\theta_2 + isin\theta_2)$ be two complex numbers. Then,

$z_1z_2 = r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)$

$$= r_1r_2[(cos\theta_1cos\theta_2 - sin\theta_1sin\theta_2) + i(sin\theta_1cos\theta_2 + cos\theta_1sin\theta_2)]$$

$$= r_1r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)] \qquad (*)$$

Therefore

$$|z_1z_2| = r_1r_2 = |z_1||z_2|$$

and

$$arg(z_1z_2) = \theta_1\theta_2 = argz_1 + argz_2$$

Thus when complex numbers are multiplied,their moduli are multiplied and their arguments are added.Also,

$$\frac{z_1}{z_2} = \frac{r_1(cos\theta_1 + isin\theta_1}{r_2(cos\theta_2 + isin\theta_2)}$$

$$\frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$$

$$|\frac{z_1}{z_2}| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$arg(\frac{z_1}{z_2}) = \theta_1 - \theta_2 = argz_1 - argz_2$$

Example: Find the moduli and the arguments of the following complex numbers.

1. $7i - 2$

   Solution:

$|7i - 2| = \sqrt{7^2 + 2^2} = \sqrt{53} = 7.28$

$arg(7i - 2) = tan^{-1}(\frac{7}{-2}) = 105.9^o$

2. $(7i - 2)(3 + 4i)$

   Solution:

   $|(7i - 2)(3 + 4i)| = |13i - 34|$

   $= \sqrt{34^2 + 13^2} = \sqrt{1325} = 36.40$

   $arg((7i - 2)(3 + 4i) = arg(13i - 34) = tan^{-1}(\frac{-13}{34}) = 159.1^o$

3. $\frac{7i-2}{3+4i}$

   Answer: $1.456, 52.8^o$

4. $(\frac{7i-2}{3+4i})^2$

   Answer: $2, 12, 105.6^o$

Example: Describe the locus of a complex number z which satisfies $|z-2| = 3|z + 2i|$.

Solution:

Put $z = x + iy$. Then

$|z - 2|^2 = 9|z + 2i|^2$

$|(x - 2) + iy|^2 = 9|x + i(y + 2)|^2$

$(x - 2)^2 + y^2 = 9[x^2 + (y + 2)^2]$

$8x^2 + 8y^2 + 4x + 36y + 32 = 0$

$x^2 + y^2 + \frac{1}{2}x + \frac{9}{2}y + 4 = 0$

$(x+\frac{1}{4})^2+(y+\frac{9}{4})^2=4+(\frac{1}{4}^2)+(\frac{9}{4})^2=\frac{18}{16}$

Locus is a circle,with center $(-\frac{1}{4},-\frac{9}{4})$ and radius $\frac{3}{4}\sqrt{2}$

De Moivre's Theorem

In general, if there are n complex numbers $z_1, z_2, ..., z_n$ with moduli $r_1, r_2, ...r_n$ and arguments $\theta_1, \theta_2, ..., \theta_n$ respectively, repeated application of equation (*) yields

$$z_1.z_2...z_n = r_1...r_n[cos(\theta_1+\theta_2+...+\theta_n)+isin(\theta_1+\theta_2+...+\theta_n)]$$

In particular if

$$z_1 = z_2 = ... = z_n = z \qquad (say)$$

$$r_1 = r_2 = ... = r_n = r \qquad (say)$$

and

$$\theta_1 = \theta_2 = ... = \theta_n = \theta \qquad (say)$$

then we have

$$z^n = r^n(cosn\theta + isinn\theta)$$

i.e

$$z^n = [r(cos\theta + isin\theta)]^n$$

$$= r^n(cosn\theta + isinn\theta)$$

$$|z^n| = |z|^n, arg(z^n) = narg(z)$$

In particular,if $r = 1$ we get Demoivre's theorem

$$(cos\theta + isin\theta)^n = cosn\theta + isin\theta$$

for any positive integer n.

This result is also valid when n is any negative integer.Suppose n is a negative integer, say $-m$ where m is a positive integer.Then

$$(cos\theta + isin\theta)^{-m} = (\frac{1}{cos\theta + isin\theta})^m = \frac{1}{(cos\theta + isin\theta)^m}$$

$$(cosm\theta + isinm\theta)^{-1} = cosm\theta - isinm\theta$$

$$= cos(-m)\theta + isin(-m)\theta$$

which shows that Demoivre's theorem is valid when n is any negative integer.

Example: Express $cos3\theta$ and $sin3\theta$ in terms of powers of $cos\theta$ and $sin\theta$ respectively.

Solution:

By Demoivre's theorem we have

$$cos3\theta + isin3\theta = (cos\theta + isin\theta)^3$$

$$= cos^3\theta + 3cos^2\theta(isin\theta) + 3cos\theta(isin\theta)^2 + (isin\theta)^3$$

$$= cos^3\theta - 3sin^2\theta cos\theta + i(3cos^2\theta sin\theta - sin^3\theta)$$

The real part of this expression then gives

$$cos3\theta = cos^3\theta - 3cos\theta sin^2\theta$$

But

$$sin^2\theta = 1 - cos^2\theta$$

Therefore

$$cos3\theta = cos^3\theta - 3cos\theta(1 - cos^2\theta)$$

$$= cos^3\theta - 3cos\theta + 3cos^3\theta$$

$$4cos^3\theta - 3cos\theta$$

and the imaginary part gives

$$sin3\theta = 3cos62\theta sin\theta - sin^3\theta$$

$$= 3sin\theta(1 - sin^2\theta) - sin^3\theta$$

$$= 3sin\theta - 3sin^3\theta - sin^3\theta$$

$$= 3sin\theta - 4sin^3\theta$$

Example: Show that if $z = cos\theta + isin\theta$ and m is a positive integer then $z^m + \frac{1}{z^m} = 2cosm\theta$

Solution:

$z = cos\theta + isin\theta$

$$z^m = (cos\theta + isin\theta)^m = cosm\theta + isinm\theta \qquad (Demoivre'stheorem)$$

$$z^{-m} = cosm\theta - isinm\theta$$

$$z^m + z^{-m} = 2cosm\theta$$

Example:

$$(z+\frac{1}{z})^5 = z^5+5z^4.\frac{1}{z}+10z^3.\frac{1}{z^2}+10z^2.\frac{1}{z^3}+5z.\frac{1}{z^4}+\frac{1}{z^5}$$
$$= z^5+5z^3+10z+\frac{10}{z}+\frac{5}{z^3}+\frac{1}{z^5}$$
$$= (z+\frac{1}{z^5})+5(z^3+\frac{1}{z^3})+10(z+\frac{1}{z})$$
$$= 2cos5\theta+2\times 5cos3\theta+2\times 10cos\theta$$
$$= 2cos5\theta+10cos3\theta+20cos\theta$$

Example: Evaluate z^8 where $z = 1+i\sqrt{3}$

Solution:

Writing z in the modulus-argument form we have $r = |z| = \sqrt{4} = 2$ and $argz = tan^{-1}\sqrt{3} = \frac{\Pi}{3}$

That is

$$z = 2(cos\frac{\Pi}{3}+isin\frac{\Pi}{3})$$

Therefore

$$z^8 = 2^8(cos\frac{\Pi}{3}+isin\frac{\Pi}{3})^8$$

By De Moivre's Theorem this becomes

$$z^8 = 2^8(cos\frac{8\Pi}{3}+isin\frac{8\Pi}{3})$$
$$= 256(-0.5+0.866i)$$
$$= -128+221.703i$$

Example: Factorize into linear factors $4z^2 + 4(1+i)z + 1 + 2i$

Solution:

$$4z^2 + 4(1+i)z + 1 + 2i = 4(z^2 + (1+i)z + \frac{1}{4}(1+2i))$$

First solve

$$z^2 + (1+i)z + \frac{1}{4}(1+2i) = 0$$

$$a = 1, b = 1+i, c = \frac{1}{4}(1+2i)$$

$$z = \frac{-1-i \pm \sqrt{(1+i)^2 - (1+2i)}}{2}$$

$$\frac{-1-i \pm \sqrt{-1}}{2} = \frac{1}{2}(-1, -i \pm i)$$

$$= -\frac{1}{2}$$

or

$$-\frac{1}{2} - i$$

$$\implies \qquad 4z^2 + (4(1+i)z + 1 + 2i = 4(z + \frac{1}{2})(z + \frac{1}{2} + i)$$

Roots of Complex Numbers

Let $z^n = \alpha, n$ a positive integer and α a complex number (**)

Theorem: (Fundamental theorem of algebra)

Every polynomial of degree at least one with arbitrary numerical coefficients has at least one root which in the general sense is complex.

Consider (**), we have

$$z^n = \alpha = r(cos\theta + isin\theta)$$

so that

$$z = r_o(cos\theta_o + isin\theta_o) \qquad provided \qquad \alpha \neq 0$$

Then by De Moivre's theorem

$$r_o^n(cosn\theta_o + isinn\theta_o) = r(cos\theta + isin\theta)$$

That is

$$z = \sqrt[n]{\alpha}, r_o^n = r, n\theta_o = \theta + 2k\Pi$$

Thus r_o is the positive nth root of r and $\theta_o = \frac{\theta \pm 2k\Pi}{n}$ has n values for $k = 0, 1, ..., n-1$ all distinct,since increasing k by unity implies increasing the argument by $\frac{2\Pi}{n}$.

The n distinct solutions of (**) are given by

$$(\alpha)^{\frac{1}{n}} = z = r^{\frac{1}{n}}(cos\frac{\theta + 2k\Pi}{n} + isin\frac{\theta + 2k\Pi}{n}) \quad k = 0, 1, ..., n-1 \qquad (***)$$

which are n distinct values of $(\alpha)^{\frac{1}{n}}$

Roots of Unity

A particular example of (**) is when $\alpha = 1$, that is if $z^n = 1, n$ is a positive integer. The roots of the equation are called nth roots of unity.Since

$1 = cos0 + isin0$

then by (***),the nth roots of unity are given by

$$1^{\frac{1}{n}} = (cos\frac{2k\Pi}{n} + isin\frac{2k\Pi}{n}), k = 0, 1, ..., n-1$$

Taking $k = 1$,the root of unity being a complex number and denoted by w is given by

$$w = cos\frac{2\Pi}{n} + isin\frac{2\Pi}{n}$$

Example: Find all the cube roots of -8

Solution:

$$\sqrt[3]{-8} = \sqrt[3]{8(cos\Pi + isin\Pi)}$$
$$= \sqrt[3]{8}(cos\frac{\Pi + 2k\Pi}{3} + isin\frac{\Pi + 2k\Pi}{3}$$

Therefore for

$$k = 0, z_0 = 2(cos\frac{\Pi}{3} + isin\frac{\Pi}{3}) = 1 + i\sqrt{3}$$
$$k = 1, z_1 = 2(cos\Pi + isin\Pi) = -2$$
$$k = 2, z_2 = 2(cos\frac{5\Pi}{3} + isin\frac{5\Pi}{3}) = 1 - i\sqrt{3}$$

Example: Solve $z^4 + 4\sqrt{3} = 4i$

Solution:

$$z^4 + 4\sqrt{3} = 4i \Longrightarrow \quad z^4 = 4i - 4\sqrt{3}$$
$$\Longrightarrow \quad z^4 = 8(cos\frac{5\Pi}{6} + isin\frac{5\pi}{6})$$

Hence using De Moivre's theorem

$$z = 8^{\frac{1}{4}}\{cos\frac{\frac{5\Pi}{6} + 2k\Pi}{4} + isin\frac{\frac{5\Pi}{6} + 2k\pi}{4}\} \qquad k = 0, 1, 2, 3$$

The four roots are

$$k = 0 : z_0 = 8^{\frac{1}{4}}(cos\frac{5\Pi}{24} + isin\frac{5\Pi}{24})$$

$$k = 1 : z_1 = 8^{\frac{1}{4}}(cos\frac{17\Pi}{24} + isin\frac{17\Pi}{24}$$

$$k = 2 : z_2 = 8^{\frac{1}{4}}(cos\frac{29\Pi}{24} + isin\frac{29\Pi}{24}) = \overline{z_0}$$

$$k = 3 : z_3 = 8^{\frac{1}{4}}(cos\frac{41\Pi}{24} + isin\frac{41\Pi}{24} = \overline{z_1}$$