STOCHASTIC PROCESS STS461

SECTION 1

OVERVIEW OF STOCHASTIC PROCESSES

COURSE CONTENTS

Random Walk, Simple and General Random Walk with absorbing and Reflecting Barriers. Markovian Properties with Finite Chains. Limit Theorem Poisson, Branching, Birth and Death Processes. Queueing Processes: M/M/1, M/M/S, M/G/1 Queues and their Waiting Time Distributions. Relevant Applications.

INTRODUCTION

Many real-world applications of probability theory have the particular feature that data are collected sequentially in time. A few examples are weather data, stock market indices, air pollution data, demographic data and political tracking polls. These also have in common that successive observations are typically not independent. We refer to any such collection of observations as a stochastic process.

DEFINITION

A stochastic process is a collection of random variables that take values in a set *S*, the state space. The collection is indexed by another set *T*, the index set. The two most common index sets are the natural numbers $= \{0,1,2,\ldots\}$, and the nonnegative real numbers $T = [0,\infty)$, which usually represent discrete time and continuous time, respectively.

The first index set thus gives a sequence of random variables (rvs) $\{X_0, X_1, X_2, \ldots\}$ and the second, a collection of random variables $\{X(t), t \geq 0\}$, one r.v. for each time *t*.

Remarks:

In general, the index set does not have to describe time but is also commonly used to describe spatial location.The state space can be finite, countably infinite, or uncountable, depending on the application.

SECTION 2

CLASSIFICATION OF GENERAL STOCHASTIC PROCESSES

2.1 INTRODUCTION

The main elements of distinguishing stochastic processes are in the nature of the statespace, the index parameter T, and the dependence relations among the random variables X_t.

2.1.1 State Space S

This is the space in which the possible values of each X_t lie. In the case that S =(0,1,2,…), we refer to the process as integer valued, or alternatively as a discrete state process.

If S = the real line $(-\infty,\infty)$, then we call X_t a real-valued stochastic process. If S is the Euclidean *k* spaced then X_t is said to be a *k*-vector process.

Remark: The choice of state space is not uniquely specified by the physical situation being described, although usually one particular choice may stand out as most appropriate.

2.1.2 Index Parameter T

If $T = (0,1,\ldots)$ then we state that $X_t$is a discrete time stochastic process. Often when T is discrete we shall write X_n instead of X_t. If $T = [0,\infty)$, then X_t is called a continuous time process.

2.2 CLASSICAL TYPE OF STOCHASTIC PROCESSES

We now describe (first briefly) then in detail, some of the classical types of stochastic processes characterized by different dependence relationships among X_t. Unless otherwise stated, we take $T = [0,\infty)$ and assume the random variables X_t are real valued.

2.2.1 Process with Stationary Independent Increments

If the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for all choices of $t_1, t_2, \ldots, t_n$ satisfying $t_1 < t_2 < \cdots < t_n$, then we say that X_t is a process with independent increments.

If the index set contains a smallest index t_0, it is also assumed$X_{t_1}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. If the index set is discrete that is T = (0,1,…), then a process with independent increments reduces to a sequence of independent random variables $Z_0 = X_0, Z_i = X_i - X_{i-1},\ i = 1,2,3,\ldots$, in the sense that knowing the individual probabilities/distributions of $Z_0, Z_1, \ldots$ enables us to determine the joint distributions of any finite set of X_i, in fact that of $X_i = Z_0 + Z_1 + \cdots + Z_i, i = 0,1,2,\ldots$

REMARKS/DEFINITIONS

1. If the distribution of the increments $X(t_1 + h) - X(t_1)$ depends only on the length h of the interval and not on the time t_1, the process is said to have stationary increments.

2. For a process with stationary increments, the distribution of $X(t_1 + h) - X(t_1)$ is the same as the distribution of $X(t_2 + h) - X(t_2)$, no matter the values of t_1, t_2 and h.

3. We now state a theorem.

   If a process $\{X_t, t \in T\}$, where $T = [0, \infty)$ or $T = (0,1,2, \ldots)$ has stationary independent increments and has a finite mean, then it is true that:

   (a) $E(X_t) = M_0 + M_1 t$ where $M_0 = E(X_0)$ and $M_1 = E(X_1) - M_0$.

   (b) $\sigma_t^2 = \sigma_0^2 + \sigma_1^2 t$ where
   $\sigma_0^2 = E[(X_0 - M_0)^2]$ and $\sigma_1^2 = E[(X_1 - M_1)^2] - \sigma_0^2$

4. Both the Brownian motion process and the Poisson process have stationary independent increments.

5. We now prove remark 3(a).

   Proof of $E(X_t) = E(X_0) + t[E(X_1 - E(X_0))]$.

   Let $f(t) = E(X_t) - E(X_0)$

   Then for any t and s, we have

   $f(t + s) = E[X_{t+s} - X_0]$

   $= E[X_{t+s} - X_s + X_s - X_0]$

   $= E[X_{t+s} - X_s] + E[X_s - X_0]$

   $= E[X_t - X_0] + E[X_s - X_0]$ using the property of stationary increments.

   The only solution to the functional equation $f(t + s) = f(t) + f(s) = f(1)t$. Differentiating with respect to t and independently with respect to s we have $f'^{(t+s)} = f'(t) = f'(s)$.

   Therefore for $s = 1$, we find $f'^{(t)} = \text{constant} = f'(t) = c$. Integrating this elementary differential equation yields $f(t) = ct + d$. But $f(0) = 2f(0)$ implies $f(0) = 0$ and therefore $d = 0$.

   The expression $f(t) = f(1)t$ is

   $E(X_t) - m_0 = (E[X_1] - m_0)t$

   $\Longrightarrow E[X_t] = m_0 + m_1 t$ as required.

2.2.2 Markov Processes

A Markov process is a process with the property that, given the value of X_t, the values of X_s, $s > t$, do not depend on the value of X_u, $u < t$; that is, the probability of any particular future behavior of the process, when the present state is known exactly, is not altered by additional knowledge concerning the past behavior (provided our knowledge of the present state is precise).

Definition1: In formal terms, a process is said to be Markov if

$$Pr\{a < X_t \leq b|X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n\}$$

where $t_1 < t_2 < \cdots t_n < t = P\{a < X_t \leq b|X_{t_n} = x_n\}$ …… (1)

Definition 2: Let A be an interval of the real line. The function

$$P(x; x; t; A) = Pr\{X_t \in A|X_5 = x\},\ t > s \qquad \text{……} \quad (2)$$

is called the transition probability function and is basic to the study of the structure of Markov processes.

We may express the condition (1) as follows:

$$Pr\{a < X_t \leq b|X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n\} = P(x_n, t_n \in A)$$

where $A = \{\xi|a < \xi \leq b\}$.

2.2.3 Martingales

Let $\{X_t\}$ be a real-valued stochastic process with discrete or continuous parameter set. We say that $\{X_t\}$ is a Martingale if, $E[|X_t|] < \infty$ for all t, and if for any $t_1 < t_2 < \cdots < t_{n+1}$, $E\left(X_{t_{n+1}}|X_{t_1} = a_1, \dots X_{t_n} = a_n\right) = a_n$ for all values of $a_1, a_2, \dots, a_n$.

2.2.4 Renewal Processes

A renewal process is a sequence T_k of independent and identically distributed (i.i.d.) positive random variables, representing the lifetimes of some "units". The first unit is placed at time zero; it fails at time T_1 and is immediately replaced by a new unit which then fails at time $T_1 + T_2$ and so on, thus motivating the name "renewal process". The time of the n^{th} renewal is $S_n = T_1 + T_2 + \cdots + T_n$.

A renewal counting process N_t counts the number of renewals in the interval $[0,t]$. Formally, $N_t = n$, for $S_n \leq t < S_{n+1}$, $n = 0,1,2,\dots$

Remark: The Poisson process with parameter λ is a renewal counting process for which the unit lifetimes have exponential distribution with common parameter λ.

2.2.5 Other examples such as Poisson process, birth and death processes and Branching Processes will be considered in small details.

SECTION 3

MARKOV CHAINS

3.1 INTRODUCTION

In this section, we consider a stochastic process $\{X_n, \ 0,1,2, \ldots\}$ that takes on a finite or countable number of possible values. Unless otherwise mentioned, this set of possible values of the process will be denoted by the set of nonnegative integers $\{0,1,2, \ldots\}$. If $X_n = i$, then the process is said to be in state i at time n. We suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j. That is we suppose that

$$P\{X_{n+1} = j | X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij} \qquad \ldots\ldots \quad (3.1)$$

for all $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$. Such a stochastic process is known as a Markov chain.

Equation (3.1) may be interpreted as stating that, for a Markov chain, the conditional distribution of any future state X_{n+1} given the past states $X_0, X_1, \ldots, X_{n-1}$ and the present state X_n is independent of the past states and depends only on the present state.

The value P_{ij} represents the probability that the processes will, when in state i, next make a transition into state j. Since probabilities are nonnegative and since the process must make a transition into some state, we have that

$$P_{ij} \geq 0, \qquad i,j \geq 0; \ \sum_{j=0}^{\infty} P_{ij} = 1, \ \ i = 0,1, \ldots$$

Let P denote the matrix of one-step transition probabilities P_{ij}, so that

$$P = \begin{vmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & & & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

Example 3.1: (Forecasting Weather)

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β. If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the preceding is a two state Markov chain whose transition probabilities are given by

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} \qquad\qquad P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

Example 3.2: (A Communications System)

Consider a communications system which transmits the digits 0 and 1. Each digit transmitted must pass through several stages, at each of which more is a probability P that the digit entered will be unchanged when it leaves. Letting X_n denote the digit entering the n$^{\text{th}}$ stage, then $\{X_n, n = 0,1,\dots\}$ is a two-state Markov chain having a transition probability matrix

$$P = \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix} \qquad P = \begin{pmatrix} P & 1-P \\ 1-P & P \end{pmatrix}$$

Example 3.3: On any given day, Gary is either cheerful (c), so-so (s) or glum (G). If he is cheerful today, then he will be c, s, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be c, s or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be c, s, or G tomorrow with probabilities 0.2, 0.3, 0.5.

Letting X_n denote Gary's mood on the n$^{\text{th}}$ day, then $\{X_n, n \geq 0\}$ is a three state Markov chain (state 0 = c, state 1 = s, state 2 = G) with transition probability matrix

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{2}{5} & \frac{1}{10} \\ \frac{2}{10} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{2} \end{pmatrix}$$

Example 3.4: (Transforming a Process into a Markov Chain)

Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

If we let the state at time n depend only on whether or not it is raining at time n, then the preceding model is not a Markov chain (why not?). However, we can transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day.

In other words, we can say that the process is in

state 0, if it rained both today and yesterday,

state 1, if it rained today but not yesterday,

state 2, if it rained yesterday but not today,

state 3, if it did not rain either yesterday or today.

The preceding would then represent a four-state Markov chain having a transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

You should carefully check the matrix *P*, and make sure you understand how it was obtained.

BIRTH AND DEATH PROCESSES

A continuous-time Markov chain with states 0, 1,… for which $q_{ij} = 0$ whenever $|i - j| > 1$ is called a birth and death process. Thus a birth and death process is a continuous-time Markov chain with states 0,1,… for which transitions from state *i* can only go to either state *i* – 1 or state *i* + 1. The state of the process is usually thought of as representing the size of some population, and when the state increases by 1, we say that a birth occurs and when it decreases by 1, we say that a death occurs.

Let λ_i and μ_i be given by

$$\lambda_i = q_{i,i+1},$$

$$\mu_i = q_{i,i-1}.$$

The values $\{\lambda_i,\ i \geq 0\}$ and $\{\mu_i,\ i \geq 1\}$ are called respectively one birth rates and one death rates. Since $\sum j\, q_{ij} = v_i$, we see that

$$v_i = \lambda_i + \mu_i$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - P_{i,i-1}.$$

Hence, we can think of a birth and death process by supposing that whenever there are *i* people in the system, the time until the next birth is exponential with rate λ_i and is independent of the time until the next death, which is exponential with rate μ_i.

EXAMPLE:

Two Birth and Death Processes

(i) **The M/M/S Queue:** Suppose that customers arrive at an s-server service station in accordance with a Poisson process having rate λ. That is, the time between successive arrivals are independent exponential random variables having mean $1/\lambda$. Each customer, upon arrival, goes directly into service if any of the servers are free, and if not, then the customer joins the queue (i.e. he waits in line). When a server finishes serving a customer, the customer leaves the system and the next customer in line, if there are any waiting, enters the service.

The successive service times are assumed to be independent exponential random variable having mean $1/\mu$. If we let *X(t)* denote the number in the system at time *t*, then $\{X(t),\ t \geq 0\}$ is a birth and death process with

$$\mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases}$$

$$\lambda_n = \lambda, \quad n \ge 0.$$

(ii) A Linear Growth Model with Immigration.

A model in which

$$\mu_n = n\mu, \quad n \ge 1,$$

$$\lambda_n = n\lambda, \quad n \ge 0,$$

is called a linear growth process with immigration. Such processes occur naturally in the study of biological reproduction and population growth. Each individual in the population is assumed to give birth at an exponential rate λ; in addition, there is an exponential rate of increase E of the population due to an external source such as immigration. Hence, the total birth rate where there are n persons in the system is $n\lambda + \theta$. Death are assumed to occur at an exponential rate μ for each member of the population, and hence $\mu_n = n\mu$.

A birth and death process is said to be a pure birth process if $\mu_n = 0$ for all n (that is, if death is impossible). The simplest example of a pure birth process is the Poisson process, which has a constant birth rate $\lambda_n = \lambda, \quad n \ge 0$.

A second example of a pure birth process results from a population in which each member acts independently and gives birth at an exponential rate λ. If we suppose that no one ever dies, then, if $X(t)$represents the population size at time t, $\{X(t), \ t \ge 0\}$ is a pure birth process with $\lambda_n = n\lambda, \quad n \ge 0$.

This pure birth process is called a Yule process.

Further Remarks about Birth and Death Process

1. We have seen that a Multiserver Exponential Queueing System is an example of birth and death process.

Description:

Consider an exponential queueing system in which there are s servers available, each serving at rate μ. An entering customer first waits in line and then goes to the first free server. This is a birth and death process with parameters

$$\mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases}$$

$$\lambda_n = \lambda, \quad n \ge 0.$$

To see why this is true, we reason as follows:

If there are n customers in the system, where $n \le s$, then n servers will be busy. Since each of these servers works at rate μ, the total departure rate will be $n\mu$. On the other hand, if there are n customers in the system, where $n > s$, then all s of the servers will be busy, and thus the total departure rate will be $s\mu$. This is known as an M/M/S queueing model.

2. For the birth and death process having parameters $\lambda_i \equiv \lambda,\ \mu_i \equiv \mu$

$$E(T_i) = \frac{1}{\lambda}\left(1 + \mu E(T_{i-1})\right)$$

where T_i denotes the time, starting from state *i*, it takes or the process to enter state $i + 1,\ i \geq 0$.

CONTINUOUS TIME MARKOV CHAINS

DEFINITIONS AND PROPERTIES

Consider a continuous-time stochastic process $\{X(t),\ t \geq 0\}$ taking on values in the set of nonnegative integers. In analogy with the definition of a discrete-time Markov chain, given earlier, we say that the process $\{X(t),\ t \geq 0\}$ is a continuous-time Markov chain if for all $s, t \geq 0$ and nonnegative integers $i, j, X(u),\ 0 \leq u \leq s$,

$$P\{X(t + s) = j | X(s) = i, X(u) = x(u), 0 \leq u \leq s\}$$

$$P\{X(t + s) = j | X(s) = i\}.$$

In other words, a continuous-time Markov chain is a stochastic process having the Markovian property that the conditional distribution of the future states at time *t* + *s*, given the present state at *t* and all past states depends only on the present state and is independent of the past. If, in addition $P\{X(t + s) = j | X(s) = i\}$ is independent of *s*, then the continuous-time Markov chain is said to have stationary or homogenous transition probabilities. All Markov chains we consider will be assumed to have stationary transition probabilities.

Suppose that a continuous-time Markov chain enters state *i* at some time, say time 0, and suppose that the process does not leave state *i* (that is, a transition does not occur) during the next *s* time units. What is the probability that the process will not leave state *i* during the following *t* time units?

To answer this, note that as the process is in state *i* at time *s*, it follows, by the Markovian property, that the probability it remains in that state during the interval [*s*, *s* + *t*] is just the (unconditional) probability that it stays in state *i* for at least *t* time units. That is, if we let $\tau_i$denote the amount of time that the process stays in state *i* before making a transition into a different state, then

$$P\{\tau_i > s + t | \tau_i > s\} = P\{\tau_i > t\}$$

for all $s, t \geq 0$. Hence, the random variable τ_i is memoryless and must thus be exponentially distributed.

The above gives us a way of constructing a continuous-time Markov chain. Namely, it is a stochastic process having the properties that each time it enters state *i*:

(i) the amount of time it spends in that state before making a transition into a different state is exponentially distributed with rate say v_i; and

(ii) when the process leaves state *i*, it will next enter state *j* with some probability, call it P_{ij}, when $\sum_{j=1} P_{ij} = 1$.

A state i for which $v_i = \infty$ is called an instantaneous state since when entered it is instantaneously left. Whereas such states are theoretically possible, we shall assume throughout that $0 \leq v_i < \infty$ for all *i*. (If $v_i = 0$, then state *i* is called absorbing since once entered it is never left).

DEFINITION:

Hence, for our purposes, a continuous-time Markov chain is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, but is such that the amount of time it spends in each state, before proceeding to the next state is exponentially distributed. In addition, the amount of time the process spends in state *i*, and the next state visited, must be independent random variables. For if the next state visited were dependent on τ_i, then information as to how long the process has already been in state *i* would be relevant to the prediction of the next state – and this would contradict the Markovian assumption.

A continuous-time Markov chain is said to be regular if, with probability 1, the n umber of transitions in any finite length of time is finite. An example of a non-regular Markov chain is the one having

$$P_{i,i+1} = 1, \quad v_i = i^2.$$

It can be shown that this Markov chain – which always goes from state *i* to *i* + 1, spending an exponentially distributed amount of time with mean $1/i^2$ in state i – will, with positive probability, make an infinite number of transitions in any time interval of length, *t, t*> 0. We shall assume from now on that all Markov chains considered are regular.

Let q_{ij} be defined by

$$q_{ij} = v_i P_{ij}, \quad \text{all } i \neq j$$

since v_i is the rate at which the process leaves state i and P_{ij} is the probability that it then goes to *j*, it follows that q_{ij} is the rate when in state *i* that the process makes a transition into state *j*; and in fact we call q_{ij} the transition rate from *i* to *j*.

Let us denote by $P_{ij}(t)$ the probability that a Markov chain, presently in state *i*, will be in state *j* after an additional time.

$$P_{ij}(t) = P\{X(t+s) = j | X(s) = i\}.$$

Definitions:

(1) q_{ij} is the transition rate from i to *j*

where $q_{ij} = v_i P_{ij}, \quad \text{all } i \neq j$

(2) v_1 is the rate at which the process leaves *i*.

SECTION 4

CHAPMAN-KOLMOGOROV EQUATIONS

The *n*-Step Transition Probabilities

By definition, the *n*-step transition probabilities P_{ij}^{n} is the probability that a process in state *i* will be in state *j* after *n* additional transitions.

(*) That is, $P_{ij}^{n} = P\{X_{n+k} = j|X_k = i\} \quad n = 0, i, j \geq 0$

The Chapman-Kolmogorov Equations

The C-K equations provide a method for computing these *n*-step transition probabilities. These equations are given by:

(**)

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m} \text{ for all } n, m \geq 0, \text{all } i, j$$

REMARKS

1. P_{ij} denotes the one-step transition probabilities

2. $P'_{ij} = P_{ij}$ from (*).

3. Observe that $P_{ik}^{n} P_{kj}^{m}$ represents the probability that starting in *i* the process will go to state *j* in *n* + *m* transitions through a path which takes it into *k* at the n^{th} transition.

Proof of C-K Equations

Using remark (3) above, we now sum over all intermediate states *k* yields the probability that the process will be in state *j* after *n* + *m* transitions. We have:

$$P_{rj}^{n+m} = P\{X_{n+m} = j|X_0 = i\}$$

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k|X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j, |X_n = k, X_0 = i\} P\{X_n = k, X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_{kj}^{n} P_{ik}^{m}$$

Matrix of *n*-step transition Probabilities: $\boldsymbol{P^{(n)}}$

Let$P^{(n)}$ denote the matrix of *n*-step transition probabilities P_{ij}^{n} then the C-K Equation given by (**) asserts that$P^{(n)+m} = P^{(n)}P^{(m)}$.

In particular,$P^{(2)} = P^{(1+1)} = P.P = P^{2}$

By induction,

$$P^{(n)} = P^{(n-1+1)} = P^{n-1}P^{1} = P^{n}$$

That is, the *n*-step transition matrix may be obtained by multiplying the matrix *P* by itself *n* times.

SOME NOTES ON MARKOV CHAIN

(1) **Irreducible Property:** We say that the Markov chain is irreducible if there is only one class – i.e. if all states communicate with each other.

CHAPMAN-KOLMOGOROV EQUATIONS AND CLASSIFICATION OF STATES

The n[th]-step transition probabilities P_{ij}^{n} is the probability that a process in state *i* will be in state *j* after *n* additional transitions, that is,

$$P_{ij}^{n} = P\{X_{n+m} = j|X_m = i\}, \quad n \geq 0, \quad i,j \geq 0.$$

Of course, $P_{ij}'^{*} = P_{ij}$. The Chapman-Kolmogorov equations provide a method for computing these *n*-step transition probabilities. These equations are:

$$P_{ij}^{n+m} = \sum_{k=0}^{x} P_{ik}^{n}P_{kj}^{m} \text{ for all } \quad n,m \geq 0, \text{ all } i,j \qquad (1)$$

and are established by observing that

$$P_{ij}^{n+m} = P\{X_{n+m} = j|X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k|X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j|\, X_n = k, X_0 = i\}\, P\{X_n = k|\, X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_{kj}^{m}\, P_{ik}^{n}$$

If we let $P^{(n)}$ denote the matrix of n-step transition probabilities P_{ij}^{n}, then Equation (1) asserts that $P^{(n)+m} = P^{(n)}.\ P^{(m)}$

where the dot represents matrix multiplication. Hence

$P^{(n)} = P\ .\ P^{(n-1)} = P.P.P^{(n-2)} = \cdots = P^{n}$,

and thus P^{n} may be calculated by multiplying the matrix P by itself n times.

Illustrative Examples

Consider Example 2.1 in which the weather is considered as a two-state Markov chain. If α = 0.7 and β = 0.4, then calculate the probability that it will rain four days from today given that it is raining today.

Solution: The one-step transition probability matrix is given by $P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

Hence, $P^{(2)} = P^{2} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

$= \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$

$P^{(4)} = \left(P^{(2)}\right)^{2} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}\begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$

$= \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}$

Hence the required probability P_{00}^{4} equals 0.5749

Example 2: Consider Example 2.4. Given that it rained on Monday and Tuesday, what is the probability that it will rain on Thursday?

Solution: The two-step transition matrix is given by

$$P^{(2)} = P^{2} = \begin{pmatrix} 0.7 & 00.3 & 0 \\ 0.5 & 00.5 & 0 \\ 0 & 0.40 & 0.6 \\ 0 & 0.20 & 0.8 \end{pmatrix}\begin{pmatrix} 0.7 & 00.3 & 0 \\ 0.5 & 00.5 & 0 \\ 0 & 0.40 & 0.6 \\ 0 & 0.20 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.49 & 0.120.21 & 0.18 \\ 0.35 & 0.200.15 & 0.30 \\ 0.20 & 0.120.20 & 0.48 \\ 0.10 & 0.160.10 & 0.64 \end{pmatrix}$$

Since rain on Thursday is equivalent to the process being in either state 0 or state 1 on Thursday, the required probability is given by $P_{00}^{2} + P_{01}^{2} = 0.49 + 0.12 = 0.61$.

STOCHASTIC PROCESS STS 461

CLASSIFICATION OF STATES

Introduction

In order to analyze precisely the asymptotic behavior of the Markov chain process, we need to introduce some principles of classifying states of a Markov chain.

Properties to be classified include: Accessible, Communicate, Aperiodic, Recurrent, Transient and Irreducible. Definitions of these properties now follow:

State j is said to be **Accessible** from state i if for some $n \geq 0, P_{ij}^n > 0$. Two states i and j accessible to each other are said to **Communicate**, and we write $i \leftrightarrow j$.

Proposition 4.2.1

Communication is an equivalence relation. That is:

(i) $i \leftrightarrow i$;

(ii) if $i \leftrightarrow j$, then $j \leftrightarrow i$;

(iii) if $i \leftrightarrow j$, and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Proof: The 1st two parts follow trivially from the definition of communication. To prove (iii) suppose that $i \leftrightarrow j$, and $j \leftrightarrow k$, then there exists *m*, *n* such that $P_{ij}^m > 0, P_{jk}^n > 0$. Hence,

$$P_{ik}^{m+n} = \sum_{r=0}^{\infty} P_{ir}^m P_{rk}^n \geq P_{ij}^m P_{jk}^n > 0$$

Similarly, we may show there exists an s for which $P_{ki}^s > 0$. Two states that communicate are said to be in the same class, and by proposition 4.2.1, any two classes are either disjoint or identical. We say that the Markov chain is **Irreducible** if there is only one class – that is, if all states communicate with each other.

State *i* is said to have period d if $P_{ii}^n = 0$, whenever *n* is not divisible by d and d is the greatest integer with this property. (If $P_{ii}^n = 0$ for all *n*> 0, then define the period of *i* to be infinite). A state with period 1 is said to be **Aperiodic**. Let *d*(i) denote the period of *i*, we can show that periodicity is a class property.

Recurrent (or Persistent): A state $i \in S$ is said to be recurrent if $Pr(\tau_i < \infty) = 1$ where τ_i is the number of steps it takes for the chain to finally visit *i*.

Transient: A state $i \in S$ is said to be transient if $Pr(\tau_i < \infty) < 1$ where τ_i is the number of steps it takes for the chain to finally visit *i*.

RANDOM WALK

Description

A one-dimensional (simple) random walk is a Markov chain whose state space is a finite or infinite subset $a, a+1, \ldots, b$ of the integers, in which the particle, if it is in state i, can in a single transition either stay in i or move to one of the adjacent states $i-1, i+1$.

Transition Matrix of a Random Walk

If the state space is taken as the nonnegative integers, the transition matrix of a random walk has the form

$$P = \begin{bmatrix} r_0 & p_0\, 0 & 0 \ldots & & \\ q_1 & r_1\, p_1 & 0 \ldots & & \\ 0 & q_2\, r_2 & p_2 \cdots & & \\ & \ddots\, 0 & q_1\, p_i & 0 & \\ & \ddots & & \vdots & \\ & & & & \end{bmatrix}$$

where $p_i > 0,\; q_i > 0,\; r_i \geq 0$ and $q_i + r_i + p_i = 1,\;\; i, 1, 2, \ldots\, (i \geq 1)$

$p_0 \geq 0, \;\; r_0 \geq 0, \; r_0 + p_0 = 1.$

Specifically, if $X_n = i$ then, for $i \geq 1$,

$$Pr\{X_{n+1} = i + 1 | X_n = i\} = P_i \colon Pr\{X_{n+1} = i - 1 | X_n = i\} = q_i$$

$$Pr\{X_{n+1} = i | X_n = i\} = r_i.$$

REMARKS:

1. We have a symmetric random walk (which is a Markov chain) if

   $$P_{i,I+1} = \frac{1}{2} = P_{i,i-1}, \;\; i = 0, \pm 1, \pm 2, \ldots$$

   That is, in each time unit the symmetric random walk is likely to take a unit step either to the left or to the right.

2. A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \ldots$ is said to be a random walk, if for some number $0 < p < 1$

   $$P_{i,i+1} = P = 1 - P_{i,i-1}, \;\; i = 0, \pm 1, \ldots$$

SOME EXAMPLES

1. The designation random walk seems apt since a realization of the process describes the path of a person (suitably intoxicated) moving randomly one step forward or backward.

2. **Gambler's Ruin**

   The fortune of a player engaged in a series of contests is often depicted by a random walk process. Specifically, suppose an individual (player A) with fortune k plays a game against an infinitely rich adversary and has probability p_k of winning one unit and probability $q_k = 1 - p_k (k \geq 1)$ of losing one unit in each contest (the choice of the contest at each stage may depend on his fortune), and $r_0 = 1$. The process $\{X_n\}$, where X_n represents his fortune after *n* contests, is a random walk. Note that once the state 0 is reached (i.e. player A is wiped out), the process remains in that state. This process is also commonly known as the "gambler's ruin".

REMARKS

(i) The random walk corresponding to $p_k = p$, $q_k = 1 - p = q$ for all $k \geq 1$ and $r_0 = 1$ with $p > q$ describes the situation of identical contests with a finite advantage to player A in each individual trial.

(ii) It can be proved that with probability $(q/p)^{x_0}$, where x_0 represent his fortune at time 0, player A is ultimately ruined (his entire fortune is lost), while with probability $1 - (q/p)^{x_0}$, his fortune increases, in the long run without limit.

3. Consider a particle which is initially at point $x = 0$ on the *x*-axis. At each subsequent time unit, it moves a unit distance to the right with probability p or a unit distance to the left with probability q where $p + q = 1$. At time *n*, let the position of a particle be X_n. This is an example of random walk.

4. A simple random walk can be described as a Markov chain $\{S_n\}$ that is such that

$$S_n = \sum_{k=1}^{n} X_k$$

where X_k are i.i.d. such that $P(X_k = 1) = P$, $P(X_k = -1) = q = 1 - p$.

This Markov chain (simple random walk) is irreducible, so it has to be either transient, null recurrent or positive recurrent and this will depend on p.

The *n*-step Transition Probabilities

We use the following proposition:

State *i* is

transient if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$$

recurrent if

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

Consider any state *i* and note that $P_{ii}^{(2n-1)} = 0$, $n = 1, 2, \ldots$ since we cannot make it back to a state in an odd number of steps. To make it back in 2*n* steps, we must take *n* steps up and *n* steps down, which has probability

$$P_{ii}^{(2n)} = (2n)P^n(1-p)^n$$

$$= \frac{(2n)!}{n!n!}\left(p(1-p)\right)^n$$

$$= \frac{\left(4p(1-p)\right)^n}{\sqrt{\pi n}}$$

where we have used Stirling's formula which says $n! \sim n^n\sqrt{n}e^{-n}\sqrt{2\pi}$.

EXPECTED VALUE AND VARIANCE OF A RANDOM WALK

Let us consider the random walk $\{X_n,\ n \geq 0\}$. We want to show that:

$$E(X_n) = n(p-q)$$

and $$V(X_n) = 4npq.$$

We make use of the following lemma:

Let Y_0 be a fixed positive integer and $\{Y_n, n \geq 1\}$ be independent and identically distributed jump variables in a random walk $\{X_n, n \geq 0\}$ such that

$$X_n = Y_0 + Y_1 + Y_2 + \cdots + Y_n \qquad (1)$$

Consider the above as a random walk, we can write equation (1) as

$$X_{n-1} = Y_0 + Y_1 + \cdots + Y_n$$

$$\Longrightarrow X_n = X_{n-1} + Y_n,\ n \geq 1 \qquad (2)$$

in general, where Y_n are i.i.d. rvs.

From Equation (2) we obtain

$$X_1 = X_0 + Y_1$$
$$X_2 = X_1 + Y_2$$

$$= X_0 + Y_1 + Y_2$$

$$\vdots$$

$$X_n = X_0 + Y_1 + Y_2 + \cdots + Y_n$$

$$E(X_n) = E\left(\sum_{k=1}^{n} Y_k\right)$$

$$= nE(Y_1) \qquad (3)$$

$$V(X_n) = Var\left(\sum_{i=1}^{n} Y_k\right)$$

$$= nVar(Y_1) \qquad (4)$$

But $E(Y_1) = 1 . p + (-1). q = p - q$.

$E(Y_1^2) = 1. p + 1. q = p + q$

$V(Y_1) = E(Y_1^2) - [E(Y_1)]^2$

$= p + q - (p - q)^2$

$= 1 - (p - q)^2 + 4pq$

$= 4pq$

Hence $E(X_n) = n(p - q)$

and $V(X_n) = 4npq$.

MORE DEFINITIONS

1. **Brownian Motion: (Definition)**

A stochastic process $\{X(t),\ t \geq 0\}$ is said to be a Brownian motion process if

(i) $X(0) = 0$
(ii) $\{X(t),\ t \geq 0\}$ has stationary and independent increments
(iii) For every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

Remarks:

(a) Wiener process is another name for Brownian process.

(b) **Definition 2:Standard Brownian Motion:**

When $\sigma = 1$, the process is called standard Brownian motion.

(c) A Martingale process is a standard Brownian motion.

Definition 3: Symmetric Random Walk

A symmetric random walk is a Markov chain with

$$P_{i,I+1} = \frac{1}{2} = P_{i,i-1},\quad i = 0, \pm 1, \ldots$$

That is, in each time unit, the symmetric random walk is likely to take a unit step either to the left or to the right.

Definition 3A: A Markov chain whose state space is given by the integers $0, \pm 1, \pm 2, \ldots$ is said to be a random walk, if, for some number $0 < p < 1$,

$$P_{i,I+1} = p = 1 - P_{i,i-1},\quad i = 0, \pm 1, \ldots$$

Remark

A Markov chain having transition probabilities

$$P_{i,I+1} = p = 1 - P_{i,i-1}, \quad i = 1,2, \ldots, N-1$$

$$P_{00} = P_{NN} = 1.$$

States 0 and *N*are called ABSORBING states since once entered they are never left.

A GAMBLING MODEL

Consider a gambler who, at each play of the game, either wins ₦1 with probability *p* or loses ₦1 with probability 1 – *p*(= *q*). If we suppose that our gambler quits playing either when he goes broke or he attains a fortune of ₦N, then the gambler's fortune is a Markov chain having transition probabilities

$$P_{i,I+1} = p = 1 - P_{i,i-1}, \quad i = 1,2, \ldots, N-1$$

$$P_{00} = P_{NN} = 1.$$

States 0 and *N* are called Absorbing states since once entered they are never left.

Remark

The above is a finite state random walk with absorbing barriers (states 0 and *N*).

Further Remarks on Random Walk

(1) $P_{00}^{2n-1} = 0, \; n = 1,2, \ldots$

(2) $P_{00}^{2n} = \binom{2n}{n} P^n (1-p)^n = \frac{(2n)!}{n!n!}\left(p(1-p)\right)^n, \; n = 1,2, \ldots$

$$P_{00}^{2n} = \sum_{i=0}^{n} \frac{(2n)!}{i!\,i!\,(n-i)!\,(n-j)!}\left(\frac{1}{4}\right)^{2n}$$

$$= \sum_{i=0}^{n} \frac{(2n)!}{n!\,n!}\frac{n!}{(n-i)!\,i!}\frac{n!}{(n-i)!\,i!}\left(\frac{1}{4}\right)^{2n}$$

$$= \left(\frac{1}{4}\right)^{2n}\binom{2n}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$$

$$= \left(\frac{1}{4}\right)^{2n}\binom{2n}{n}\binom{2n}{n} \qquad (1)$$

where the last equality uses the combinatorial identity

$$\binom{2n}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} \qquad (2)$$

Now,

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$$

$$\cong \frac{(2n)^{2n+1/2}e^{-2n}\sqrt{2\pi}}{n^{2n+1}e^{-2n}(2\pi)}$$

$$\cong \frac{4}{\sqrt{\pi n}}$$

From (1), we see that $P_{00}^{2n} \cong \frac{1}{\pi n}$ which shows that $\sum_n P_{00}^{2n} = \infty$, and thus all states are recurrent (or persistent).

6 Queueing Processes

6.1 INTRODUCTION

A Queueing Process: Is a process in which customers arrive at some designated place where a service of some kind is being rendered. It is assumed that the time between arrivals, or inter-arrival time, and the time that is spent in providing service for a given customer are governed by probabilistic laws.

It is assumed in this system/model that customers upon arrival are made to wait queue until it is their turn to be served. Once served they are generally assumed to leave the system.

In this system, we will be interested in determining, among other things, such quantities as the average number of customers in the system (or in the queue) and the average time a customer spent in the system (or spent waiting in the queue).

6.2 TYPES OF QUEUEING SYSTEM

We will consider queueing systems in which all of the defining probability distributions are assumed to be exponential. For instance, the simplest of such model is to assume that customers arrive in accordance to a Poisson process and (thus the inter-arrival times are exponentially distributed) and are served one at a time by a single server who takes an exponentially distributed length of time for each service. These exponential queue models are special examples of continuous-time Markov chain. Specifically, we describe the following queueing models.

6.2.1 The Queueing System M/M/1: In this system, the first M refers to the fact that the inter-arrival process is Markovian (since it is a Poisson process) and the second M refers to the fact that the service distribution is exponential (and, hence, Markovian). The 1 refers to the fact that there is a single server.

Theory: Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate λ. That is, the times between successive arrivals are i.i.d. exponential random variables having mean $1/\lambda$. Service times are i.i.d. exponential with rate μ and independent of the arrivals (note that μ does not denote the mean here; the mean service time is $1/\mu$). If the server is busy, incoming customers will wait in line and as soon as a service is completed, the next begins.

If we let X(t) denote the number in the system at time *t* then $\{X(t),\ t \geq 0\}$ is a birth and death process with

$$\mu_n = \mu, \quad n \geq 1$$

$$\lambda_n = \lambda, \quad n \geq 0$$

6.2.2 The Queueing System M/M/S: This is a multiserver exponential queueing system. Consider an exponential queueing system in which there are *S* servers available, each serving at rate μ. An entering customer first waits in line and then goes to the first free server. This is a birth and death process with parameters

$$\mu_n = \begin{cases} n\mu, & 1 \leq n \leq s \\ s\mu, & n > s \end{cases}$$

$$\lambda_n = \lambda, \qquad n \geq 0$$

6.2.3 The Queueing System M/D/1: This is an example of a non-Markovian queueing system. In this system, the service times are deterministic. Also, both interarrival times and service times have some general distribution *D.*

6.2.4 The Queueing System G/G/1: In this system which is non-Markovian, both interarrival times and service times have some general distribution not necessarily exponential.

6.3 PROPERTIES OF QUEUEING SYSTEM

In this section, we consider and illustrate with an example the stationary distribution of a queueing system. When a queueing system has a stationary distribution, it is said to be in equilibrium. Let us illustrate this concept for M/M/1 queueing system.

Example

Consider the M/M/1 queue with $\rho < 1$ in equilibrium

What is the expected number of customers in the system?
What is the expected queue length?
When a customer arrives, what is the probability that she does not have to wait in line?
When a customer arrives, what is her expected waiting time until service?
When a customer arrives, what is her expected total time in the system?

Let us introduce some random variables. Thus, let

N = the number of customers in the system

Q = the queue length

W = the waiting time until service

T = the total time spent in the system

For (a), we know that N has distribution

$$\pi_k = (1-\rho)\rho^k, \qquad k = 0,1,\dots$$

the geometric distribution including 0 with success probability $1-\rho$, and we know that

$E(N) = \frac{\rho}{1-\rho}$ which answers (a)

ρ is the traffic intensity
Equilibrium distribution $\{\pi_n\} =$ is given by $\pi_n = (1-\rho)\rho^n,\ n \geq 0,\ \ n = 0,1,\dots$

(b) Note that

$$Q = \begin{cases} 0 & \text{if } N = 0 \\ N-1 & \text{if } N \geq 1 \end{cases}$$

and hence

$$P(Q = 0) = \pi_0 + \pi_1$$

$$P(Q = k) = \pi_{k+1}, k \geq 1$$

which gives

$$E[Q] = \sum_{k=0}^{\infty} kP(Q = k) = \sum_{k=1}^{\infty} k\pi_{k+1}$$

$$= \rho\sum_{k=1}^{\infty} k(1-\rho)\rho^k = \rho E[N] = \frac{\rho^2}{1-\rho}$$

The answer to (c) is simply $\pi_0 = 1 - \rho$

For (d), note that $W = 0$ if the system is empty and the sum of μ i.i.d. exponential. With mean $1/\mu$ if there are N customers in the system (keep in mind that μ does not denote the mean but the service rate).

$$E[W] = E[N]\frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$$

For (e), let S be a service time and note that $T = W + S$ to obtain

$$E[T] = \frac{\rho}{\mu(1-\rho)} + \frac{1}{\mu} = \frac{1}{\mu(1-\rho)}$$