ON THE FLOW OF A POWER LAW FLUID OVER A FLAT PLATE IN THE PRESENCE OF A PRESSURE GRADIENT

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ABSTRACT

We study the steady pseudoplastic flow of a power law fluid over a flat plate in the presence of a pressure gradient. The pressure gradient increases along the flow direction. We examine for $u\left(x\right)=U_o\, x^{\frac{\alpha-1}{2}}$, the flow parallel to the flat plate, the appropriate condition for a similarity solution. The flow has a unique solution when the power law exponent varies between ½ and 1 and the result showed that the power law exponent has appreciable influence on the flow.

Keywords: Fluid dynamics, power law fluid, pressure gradient.

1 INTRODUCTION

The study of non – Newtonian fluid has been of much interest to scientist because some industrial materials are non – Newtonian. Of particular interest is power law fluid for which the shear stress τ, is given by

$$\tau_{yx} = -m \left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y},$$

where m is the flow index, $\frac{\partial u}{\partial y}$ is the shear rate and n is the power law exponent.

When n is less than 1, the fluid is pseudoplastic, for n equal to 1, the fluid is Newtonian and when n greater than 1, the fluid is dilatant. Some examples of a power law fluid are cement rock in water, napalm in kerosene, lime in water, Illinois yellow clay in water.

Many researchers have shown great interest in the study of the flow of Newtonian and non-Newtonian fluids. Bird (Bird 1959) examined the flow of an unsteady pseudoplastic fluid near a moving wall, he gave interesting results for n = 5/6, 2/3, ½ and 1/3. Kelly *et al*. (Kelly *et al.* 1999) studied a non- conventional fluid dynamic problem by means of the boundary layer approximation. The authors find similarity solution for the continuity, momentum, and energy and diffusion equation in a closed form in terms of the exponential functions for the temperature and concentration fields. A large part of the work is devoted to the analysis of the result; the asymptotic behaviour of the solutions for small and high values of the non-dimensional numbers that govern the energy and diffusion equation (i.e. prandtl and Schmidt number) is discussed.

Talhouk (Talhouk 1999) studied flow of visco-elastic weakly compressible fluids having a differential constitutive equation. The main goals were to prove the existence and uniqueness of the solution to the Jeffery's and maximal type of constitutive equations respectively. In both cases, they showed that when the compressibility goes to zero, the corresponding weakly compressible steady solution goes to the incompressible one.

Fetecau (Fetecau 1999) investigated the existence and uniqueness of unidirectional spherical gap flows of the simple fluid of integral type of the first order and of the fluid of second grade, are proved. To this purpose some results from the theory of abstract Volterra integro-differential equations and from the theory of abstract linear pseudo parabolic equations are used.

Marusic-Paloka (Marusic-Paloka and Edward 2001) examined the steady flow of a dilatants non-Newtonian fluid obeying the power law in unbounded channels and pipes. A proof of existence and uniqueness of the solution for Leary's problem for such a fluid is given as well as the delay estimate for the solution. For the existence result, he applies Garlekins procedure using monotonicity of the principal of the operator and the continuity of the inertia term.

Bloom and Hao (Bloom and Hao 2001) considered a non-Newtonian model for the equations of motion of a bipolar fluid. The case of flow in unbounded channel is treated. The existence of solutions is proved by considering a sequence of approximate solutions in some bounded sub-domains, and then by showing that there exist subsequences of approximate solutions whose limit is a solution of the problem.

Marusic-Paloka (Marusic-Paloka and E 2001) discussed the problem of a purely viscous flow of a non-Newtonian fluid obeying the power law in an exterior domain. It is proved that for pseudo plastic fluids the stoke paradox never appears while for the dilatants ones, it appears even for three – dimensional exterior problems.

Hassanien *et al.* (Hassanien *et al.* 1998) discussed the flow and heat transfer in a power law fluid over a non-isothermal stretching sheet. They presented a boundary layer analysis for the problem of flow and heat transfer from a power law fluid to a continuous stretching sheet with variable wall temperature. They performed parametric studies to investigate the effect of non-Newtonian flow index, generalized Prandtl number, power law surface temperature and surface mass transfer rate. Their results showed that friction factor and heat transfer depend strongly on the flow parameters.

Elena and Paola (Elena and Paola 1998) investigated the flow of a Bingham fluid in contact with a Newtonian fluid. The Newtonian fluid played the role of a lubricant. A free boundary problem coupled by means of diffraction conditions with a boundary value problem of parabolic type was obtained. They examined the steady state solution and a regular model related to the appearance of a new rigid zone.

Howell *et al.* (Howell *et al.* 1997) examined momentum and heat transfer on a continuously moving surface in a power law fluid. They examined the momentum and heat transfer occurring in the laminar boundary layer on a continuously moving and stretching two dimensional surface in non-Newtonian power law fluid. Their results include situations when the velocity is non linear and when the surface is stretching linearly.

In this paper, we discuss the steady pseudoplastic flow of a power law fluid over a flat plate in the presence of a pressure gradient. The pressure gradient increases along the flow direction. We examine for $u(x) = U_o \; x^{\frac{\alpha-1}{2}}$, the flow parallel to the flat plate, the appropriate condition for similarity solution. The flow has a unique solution when the power law exponent varies between ½ and 1 and the result showed that the power law exponent has appreciable influence on the flow.

2 MATHEMATICAL FORMULATION

We consider the pseudoplastic flow of a power law fluid over an infinite plate in the presence of the pressure gradient with a uniform flow of constant physical properties including density. The relevant governing equations for the steady flow are as follows.

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum Equation

$$\rho\left(u\frac{\partial u}{\partial v} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} \tag{2}$$

where,

$$\tau_{yx} = -m\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{3}$$

With boundary and initial conditions

$$u(x,0) = U, \quad u(x,\infty) = 0 \tag{4}$$
$$\frac{\partial u}{\partial y}(x,0) = U.$$

where ρ is the density,

u is the x- component velocity,

v is the y- component velocity,

τ_{yx} is the stress,

p is the pressure.

3 METHOD OF SOLUTION

Assume $\frac{\partial u}{\partial y}$ is every where negative, (see Bird (Bird 1959)).

Equation (3) becomes,

$$\tau_{yx} = m\left(-\frac{\partial u}{\partial y}\right)^{n} \tag{5}$$

And so, equation (2) becomes

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - m\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^{n} \tag{6}$$

Introducing the stream function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{7}$$

the continuity equation (1) is automatically satisfied.

If the flow parallel to the flat plate is given by

$$u(x) = U_o x^{\frac{\alpha-1}{2}}, \tag{8}$$

then, the pressure gradient along the plate can be computed from Euler's equation, so that

$$\frac{\partial p}{\partial x} = -\rho\left(\frac{\alpha-1}{2}\right)U_o^2 x^{\alpha-2} \tag{9}$$

In a region of decreasing pressure along the flow direction, the net pressure force acting on the fluid tends to accelerate it; the pressure gradient is called favourable. In a region in which pressure increases along the flow direction; it is called an adverse pressure gradient.

Equation (9) implies that, for pseudoplastic flow in the presence of an adverse pressure gradient $\alpha < -1$; $\frac{\partial p}{\partial x} \geq 0$.

Using equations (7) and (9) in equation (6) and define a similarity variable

$$\eta = Ay\, x^{\frac{\alpha-1}{2}} \tag{10}$$

Such that

$$\psi\ (\mathrm{x},\mathrm{y}) = \mathrm{D}\, f\ (\eta) \tag{11}$$

where A and D are constants.

Equation (6) together with the boundary and initial conditions (4) become,

$$A^2\ \mathrm{D}^2 \left(\frac{\alpha - 1}{2}\right) f'^2 = \left(\frac{\alpha - 1}{2}\right) \mathrm{U}_\mathrm{o}^2\, Vn\ \mathrm{A}^{2\mathrm{n}+1}\ \mathrm{D}^\mathrm{n} \left(-f''\right)^{n-1} f''' \tag{12}$$

$$f(0) = 1,\quad f'\ (0) = 1,\quad f'\ (\infty) = 0 \tag{13}$$

REMARK 1: *Similarity solution exist for*

$$\alpha = \frac{2n-3}{2n-1} \tag{14}$$

4 EXISTENCE AND UNIQUENESS OF SOLUTION

THEOREM 1: For every ½ < n < 1, 0 < y < ∞, problem (6) satisfying the boundary and initial conditions (4) has a unique solution.

REMARK 2: *To prove the existence and uniqueness of solution of problem (6) satisfying conditions (4), it is sufficient to prove the existence and uniqueness of solution of problem (12) satisfying conditions (13).*

THEOREM 2: There exist a solution of problem (12) satisfying conditions (13) for every ½ < n < 1.

Proof:

For pseudoplastic flow with adverse pressure gradient $\alpha < -1$, this implies,

$$\alpha = \frac{2\mathrm{n} - 3}{2\mathrm{n} - 1} < -1 \tag{15}$$

and so,

$$(2n-1)\ (\mathrm{n} - 1) < 0\,. \tag{16}$$

Thus

$$½ < \mathrm{n} < 1 \tag{17}$$

This completes the proof.

Now, resolve equation (12) into system of equations.

Let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f & (\eta) \\ f' & (\eta) \\ f'' & (\eta) \end{pmatrix} \qquad (18)$$

We consider,

$$\begin{pmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \end{pmatrix} \quad \begin{pmatrix} x_2 \\ x_3 \\ \dfrac{A^2 \ \mathrm{D}^2 \left(\dfrac{\alpha - 1}{2}\right) x_2^2 \ - \left(\dfrac{\alpha-1}{2}\right) \ U_0{}^2}{Vn \ \mathrm{A}^{2n+1} \ D^n \left(-\mathrm{x}_3\right)^{\mathrm{n-1}}} \end{pmatrix} \qquad (19)$$

Together with the initial conditions.

$$\begin{pmatrix} x_1 & (0) \\ x_2 & (0) \\ x_3 & (0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -\beta \end{pmatrix} \qquad (20)$$

$$0 \leq \eta < \infty \ , \ 1 \leq \mathrm{x}_1 < \mathrm{L}, \ 0 \leq \mathrm{x}_2 \leq 1, \ -\beta \leq \mathrm{x}_3 \leq 0,$$

Where L, β are positive constants. The initial condition (20) is obtained from the initial conditions (13) using the system of equation (18).

Problem (19) can be written as

$$\begin{pmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \end{pmatrix} = \begin{pmatrix} f_1 & (\eta, \ x_1, \ x_{2,} \ x_3 \ ) \\ f_2 & (\eta, \ x_1, \ x_{2,} \ x_3 \ ) \\ f_3 & (\eta, \ x_1, \ x_{2,} \ x_3 \ ) \end{pmatrix} \qquad (21)$$

THEOREM 3: For every ½ < n < 1 and for which conditions (20) hold, $f_1 \ (\eta_1 \ x_1, \ x_{2,} \ x_3 \ )$ in problem (21) are Lipschitz continuous.

Proof:

Consider the partial derivative

$$\frac{\partial f_i}{\partial x_j} \quad i, j = 1, 2, 3 \tag{22}$$

Clearly,

For every ½ < n < 1 and $0 < \eta < \infty$, $1 < x_1 < L$, $0 \leq x_2 \leq 1$, $-\Gamma \leq x_3 < 0$,

$\frac{\partial f_i}{\partial x_j}$ i, j = 1, 2, 3 is bounded

And so, there exist a constant K > 0, such that

$$\text{Max} \frac{\partial f_i}{\partial x_j} \leq K \tag{23}$$

Hence $f_1 \left(\eta \;\; x_1, \; x_{2,} \;\; x_3 \right)$, i = 1, 2, 3 are Lipschitz continuous.

This completes the proof.

Re-write problem (21) as

$$X' (\eta) = F (\eta, X) \tag{24}$$

Where

$$X = (x_{1,} \quad x_2 \quad x_3) \tag{25}$$

And

$$F = (f_1, \; f_2, \; f_3) \tag{26}$$

Also the initial condition (20) is written as,

$$X(0) = B \tag{27}$$

THEOREM 4:

Let X (0) = B be a point in an open subset of A of $R^1 \;\; x \;\; R^3$ suppose that;

(i) $F (\eta, X)$ is continuous in A and

(ii) F satisfies lipschitz condition of the form $\| F (\eta, X) - F (\eta, Y) \| \leq K \| X - Y \|$

For some real K and all points (η, X) and (η, Y). Then a unique solution of problem (24) satisfying initial condition (27) exist in the interval $O \leq \eta \leq \infty$

Proof:

Since

$$X' = F(\eta, X) \tag{28}$$

And

$$X(0) = B \tag{29}$$

Then,

$$X(\eta) = B + \int_0^{\eta} F(s, X(s))\, ds \tag{30}$$

Let M denotes the set of continuous functions from R^1 to R^3

Define the norm of any function X in M as

$$\|X\|_M = \text{Sup} \; \| X(\eta) \| \tag{31}$$

We shall show that equation (31) the norm M is Banach space.

Clearly, equation (31) is a norm on M. Suppose M is complete.

Let $\{X_n\}$ $n \in N$ be a Cauchy sequence in M. then, given $\in > 0$, there exist $n_0 \in N$

Such that

$$\|X_n - X_m\|_M < \in, \; \mathit{If} \; n, m \geq n_0 \tag{32}$$

$$\|X_n - X_m\|_M = \text{Sup} \; \| X_n(\eta) - X_m(\eta) \| < \in \tag{33}$$

$\{X_n\}$ $n \in N$ is Cauchy sequence.

Now, define $X \varepsilon M$ by

$$X(\eta) = \lim_{n \to \infty} X_n(\eta) \tag{34}$$

We show that

(i) $X \varepsilon M$ and (ii) $\|X_n - X\| \to 0$ (35)

Now fix $n \geq N$, by (34)

If given $\in' > 0$ there exist such that

$$\|X_q(\eta) - X(\eta)\| \leq \in' \tag{36}$$

Let

m = max (N, q), then,

$$\|X_n(\eta) - X(\eta)\| \leq \|X_n(\eta) - X_q(\eta)\| + \|x_\eta(\eta) - X_n\| \leq \in + \in' \tag{37}$$

Since $\in' 0$ can be arbitrarily small, it implies that

$$\|X_n(\eta) - X(\eta)\| \leq \in \text{ for all } n \geq N \tag{38}$$

$$\|X(\eta)\| \leq \in + \|X_n(\eta)\| \tag{39}$$

X_n is bounded being element M so, therefore, X is bounded and such is in M.

Again from (34),

$$\|X_n - X\|_m = \sup \|X_n(\eta) - X(\eta)\| \leq \in \qquad \text{for all } n \varepsilon N. \tag{40}$$

$$X_n \to X \text{ and } \|X_n - X\| \to 0. \tag{41}$$

Hence, M is complete and so norm M is a Banach space

We now, define the set $M(\sigma)$ as

$$M(\sigma) = \{X \in M : \|X\|_m \leq \sigma\}. \tag{42}$$

And operator U by

$$(UX)(\eta) = B + \int_v^\eta F(s, x(s))\,ds \tag{43}$$

for $X \in M(\sigma)$

$$\text{Where } \sigma = \|B\| + t\,\delta \tag{44}$$

t being the bound for $\|F(\eta, X)\|$

From (43),

$$\|(UX)(\eta) - B\| - \left\|\int_0^\eta F(s, x(s))\,ds\right\| \leq t\,\delta \tag{45}$$

This implies

$$\| \ (U\mathrm{X})(\eta)\text{-}\ \mathrm{B}\ \| \leq \ \mathrm{t}\delta \tag{47}$$

Therefore

$$\| \ (U\mathrm{X})(\eta) \ \| \leq \ \| \mathrm{B} \ \| + \mathrm{t}\delta \ = \ \sigma \tag{48}$$

Hence, U maps M (σ) into itself

Let X, Y ε M (σ) and δ K < 1.

$$\begin{aligned} \| \ (U\mathrm{X})\ (\eta) - (U\mathrm{Y})\ (\eta) \ \| &= \ \| \int_0^\eta \ \mathrm{F}(\mathrm{s},\ \mathrm{x}\ (\mathrm{s})\,) - \mathrm{F}(\mathrm{s},\mathrm{y}\ (\mathrm{s})\,)\ \mathrm{ds} \ \| \\ &\leq \delta \sup \ \| \ \mathrm{F}(\mathrm{s},\ \mathrm{x}\ (\mathrm{s})\,) - \mathrm{F}(\mathrm{s},\mathrm{y}\ (\mathrm{s})\,) \| \ \mathrm{ds} \\ &\leq \ \delta\, \mathrm{K} \sup \| \ \ \mathrm{X}(\mathrm{s}) - \mathrm{Y}(\mathrm{s}) \ \ \| \end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned} \| UX - U\mathrm{Y} \| &= \sup \ \| (UX)(\eta)\text{-}(U\mathrm{Y})(\eta) \ \| \\ &\leq \delta \ \ \mathrm{K} \sup \| \mathrm{X} \ \ (\mathrm{s}) - Y\, s) \ \| \\ &= \delta\, \mathrm{K} \ \ \| X - \mathrm{Y} \ \| \end{aligned} \tag{50}$$

Since, $\delta\,\mathrm{K} < 1$. U is a contraction mapping and contraction U has a fixed point in M. Thus, there is a unique function in M which is a solution of problem (24). And since any solution of (24) is in M and for sufficiently small, there exist a unique solution of problem (24).

Proof of Theorem 1

The existence and uniqueness of problem (24) implies that problem (21) has a unique solution, and so, problem (19) has a unique solution. Since problem (19) has a unique solution. Therefore, the problem (12) satisfying condition (13) has a unique solution. Hence, the problem (6) satisfying condition (4) has a unique solution.

5 NUMERICIAL SOLUTION

Resolve,

$$A^2 D^2 \left(\frac{\alpha - 1}{2} \right) (f')^2 = \left(\frac{\alpha - 1}{2} \right) \mathrm{U}^2{}_0 + \upsilon\, n A^{2n+1} D^n \left(- f'' \right)^{n-1} f''' \tag{51}$$

Together with boundary and initial conditions

$$f \ (0) = 1, \ \ f'(0) \ = 1 \,, \ \ f'(\infty) = 0 \tag{52}$$

Into system of equations as follows

Let,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \eta \\ f \\ f' \\ f'' \end{pmatrix} \tag{53}$$

We consider,

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 1 \\ y_3 \\ y_4 \\ \dfrac{(\alpha-1)\left(A^2 D^2 y_3^{\,2} - U_0^{\,2}\right)}{2\ \upsilon\, n\, A^{2n+1} D^n (-y_4)^{n-1}} \end{pmatrix} \tag{54}$$

Together with the initial conditions

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -\beta \end{pmatrix} \tag{55}$$

Note: *β is guessed such that boundary and initial conditions (52) is satisfied. Problem (54) together with condition (55) is solved numerically and the result is presented as the velocity profile in figure 1.The power law exponent n = 0.6, 0.7, 0.8 and 0.9. The kinematics viscosity υ = 0.5. The flow rate is a function of n (see equations (8) and (9)).*

6 DISCUSSION OF RESULT AND CONCLUSION

In this work, the mathematical model for the flow of a power law fluid over a flat plate in the presence of a pressure gradient is presented. The existence and uniqueness theorems 1, 2, 3 and 4 were used to establish that the mathematical model has a unique solution when the power law exponent varies between ½ and 1, since the mathematical model has a unique solution; therefore it represents a real life problem. The numerical result is presented in figure 1 as the velocity profile. The figure showed that the rate of the flow increases with increases in the value of the power law exponent. The increase in the power law exponent cause increase in the pressure gradient, and so, for the flow of a power law fluid in the presence of an adverse pressure gradient the flow rate increase with increase in the value of the power law exponent. Hence, the power law exponent has appreciable influence on the flow.

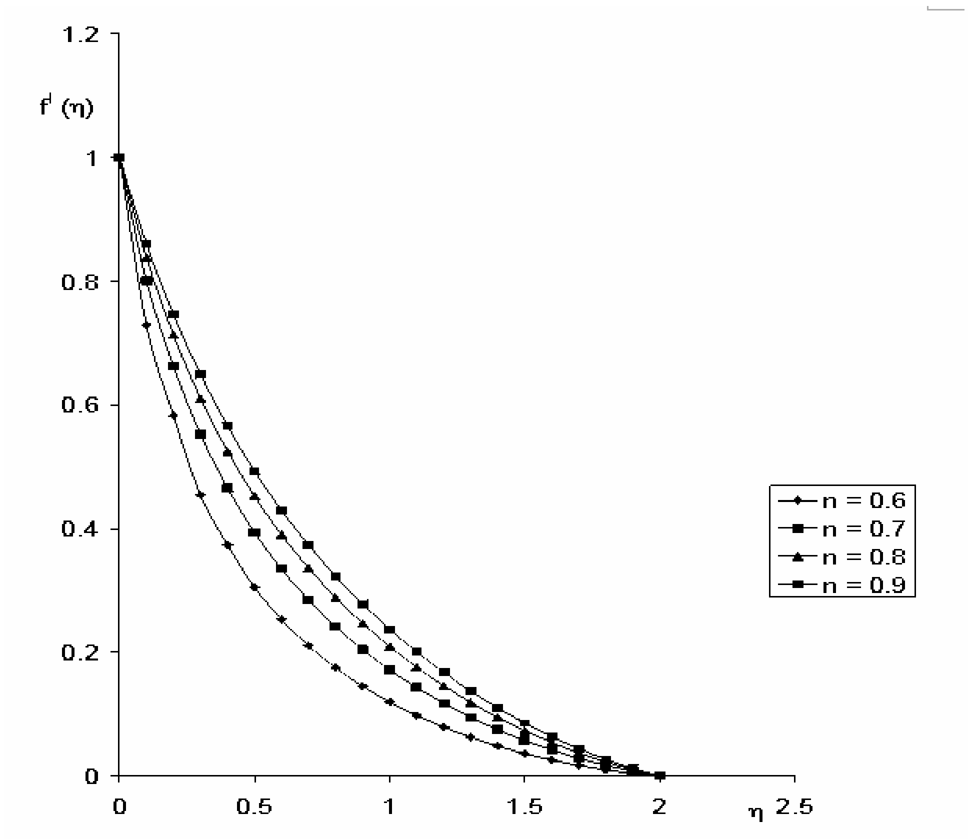

Figure 1: Velocity Profile of flow of a power law fluid over a flat plate in the presence of a preesure gradient

The fact that $f'(\eta) \to 0$ as $\eta \to \infty$ implies that this adverse pressure gradient together with the action of the shear force, if they act for a sufficient length , bring the boundary layer to rest and the flow separates from the surface. This flow separation has serious consequences in production engineering which involves the transportation of production materials (fluids) over a surface from one point of production to another, due to the continuing action of the adverse pressure gradient downstream of separation point, reversed flow is formed which acts to increase drastically the drag force acting in the surface. Hence, in designing of flow producing surfaces the onset of separation should be avoided. In fact, the present work has many other applications in engineering processes.

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