DEPARTMENT OF MATHEMATICS
UNIVERSITY OF AGRICULTURE, ABEOKUTA
2009/2010 SECOND SEMESTER EXAMINATIONS
MTS 314 THEORY OF MODULES TUE OCTOBER 5, 2010

INSTRUCTION: ANSWER **ALL** QUESTIONS

Time ALLOWED: $2\frac{1}{2}$ HOURS

1. (a) Let R be any ring and let $A = R \times \mathcal{Z}$ where $\mathcal{Z}$ is the set of integers. Define + and $\bullet$ in A by:

$$\begin{aligned}(a,m)+(b,n) &= (a+b, m+n),\\ (a,m)\bullet(b,n) &= (ab+na+mb, mn), \forall\ a,b \in R \text{ and } m,n \in \mathcal{Z}.\end{aligned}$$

   i. Show that $(A,+,\bullet)$ is a ring with (0,1) as the unity.
   ii. If $\phi : R \to A$ is a mapping defined by $\phi(a) = (a,0)\ \forall\ a \in R$, show that ϕ is an injective homomorphism.

   (b) i. Let R be any ring with unity and for each $a \in R$, let there exist $x \in R$ such that $a^2x = a$. Show that $ax = xa$ and also show that ax and xa are idempotents in Z(R), the center of R.
   ii. Let $B = \{0,2,4,6,8\}$. Show that B is a subring of $\mathcal{Z}_{10}$, the ring of integers modulo 10 with unity different from the unity of $\mathcal{Z}_{10}$ and state the unity of B.
   iii. Let R be a commutative ring and let a and b be nilpotent elements of R. Show that $(a+b)$ is also nilpotent.

   (c) Let R be a ring and let I be a subset of R. Let

$$\begin{aligned}r(I) &= \{r \in R : Ir = 0\} \text{ and}\\ l(I) &= \{r \in R : rI = 0\}.\end{aligned}$$

   i. Show that r(I) and l(I) are right and left ideals of R respectively.
   ii. Given that A is an ideal in R, show that r(I) and l(I) are ideals in R.

   (d) If $R = \mathcal{Z}$ and $I = (42)$, $J = (132)$ are ideals of R, compute (I:J), the ideal quotient of I and J.

2. (a) i. Let I be an ideal of R and define the multiplication map $* : [R/I] \times R \to R/I$ by

$$*(m+I, n) = mn + I.$$

   Show that R/I is a right R-module.
   ii. Show that the direct product of two distinct R-modules is also an R-module.
   iii. Let $(M_i)_{i\in I}$ be a family of R-submodules of an R-module M. Show that $\bigcap_{i\in I} M_i$ is also an R-submodule.

iv. Let M be an R-module and for $m \in M$, let K be a set defined by

$$K = \{rm + nm : r \in R, n \in \mathcal{Z}\}.$$

Show that K is an R-submodule of M.

(b) i. Let M be an R-module and let r be some fixed element of R. Show that the mapping $f : M \to M$ defined by $f(m) = rm \; \forall \; m \in M$ is an R-homomorphism.

ii. Let A and B be R-submodules of R-modules M and N respectively. Show that

$$[M \times N]/[A \times B] \cong [M/A] \times [N/B].$$

(c) Define the following:

i. Exact sequence

ii. Short exact sequence

iii. Split exact sequence.

iv. Cokernel

v. Coimage

(d) Draw a commutative diagram of R-modules with exact rows and columns.

(a) i. Let U and V be vector spaces over the field F. Show that

$$\mathrm{Hom}_F(U, V) \cong F^{m \times n}.$$

ii. Compute the rank of the linear mapping $\phi : \mathcal{R}^5 \to \mathcal{R}^4$ given by

$$\phi(a, b, c, d, e) = (2a+3b+c+4e, 3a+b+2c-d+e, 4a-b+3c-2d-2e, 5a+4b+3c-d+6e).$$

(b) Let $A = \begin{bmatrix} -x & 4 & -2 \\ -3 & 8-x & 3 \\ 4 & -8 & -2-x \end{bmatrix}$ be a given matrix. Compute:

i. the invariant factors of A over the ring $\mathcal{Q}[x]$,

ii. the rank of A.

(a) Let $B = \{\sin x, \cos x, \sin 2x, \cos 2x\}$ and $V = \mathrm{span}(B)$. In the space of all continuous functions on $\mathcal{R}$, V is a four-dimensional subspace with basis B. Define $\phi : V \times V \to \mathcal{R}$ by

$$\phi(f, g) = f'(0) \bullet g''(0).$$

Show that ϕ is a bilinear form on V and compute its matrix representation wrt the basis B.

(b) Let q be the quadratic form associated with the symmetric bilinear form f. Show that:

i. $2f(u, v) = q(u + v) - q(u) - q(v)$,

ii. $4f(u, v) = q(u + v) - q(u - v)$.

(c) Reduce the quadratic polynomial

$$\begin{aligned} q(a, b, c, d) &= a^2 + 2ab + 2b^2 + 6c^2 - 4ac - 10bc \\ &\quad +11d^2 - 6ad - 2bd + 18cd \end{aligned}$$

to a diagonal form and state its rank and signature.