UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA
B.Sc.(Hons) MATHEMATICS DEGREE EXAMINATION 2009/2010
FIRST SEMESTER

MTS 411 - ADVANCED ALGEBRA I

JUNE 2010
TIME ALLOWED: $2\frac{1}{2}$

INSTRUCTION(S): Attempt any FOUR (4) Questions
(All rings are assumed to be commutative)

1. (a) Let $f : R_1 \longrightarrow R_2$ be a ring homomorphism. The kernel I of f is an ideal of R_1 and the image C of f is a subring of R_2. Show that quotient ring R_1/I is isomorphic to C.
   (b) Let I be an ideal of a ring R. Show that there is a bijection between the set of all ideals J of R such that $I \subset J$ and the set of all ideals R/I such that $\{J : I \text{ an ideal of } R, I \subset J\} \longrightarrow \{K : K \text{ an ideal of } R/I\}, J \longrightarrow J/I$
   (c) Prove that any non-zero ring R is field if and only if it has exactly two different ideals (0) and (1)

2. (a) Let N,K be R-submodules of an R-module M. A map $f : N \oplus K \longrightarrow N + K$ is defined by $f((n,k) = n+k$ is a surjective R-module homomorphism whose kernel is R-isomorphic to the submodule $N \cap K$. Prove that $N \oplus K$ is isomorphic to $N + K$ if $N \cap K = \{0\}$
   (b) i. Prove that the module $R^n \oplus_{1\leqslant i\leqslant n} R$ is a free R-module of rank n
   ii. Show that every free R-module of rank n is isomorphic to R^n.
   (c) Let R be a ring and M and R-module, show that $M \otimes_R R \simeq M$

3. (a) When is a R-module M called a Noetherian module?
   (b) Let R be a ring and I an ideal of R. If R/I is a Noetherian R-module, show that R/I is a Noetherian ring.
   (c) Let M be an R-module and N a submodule of M. Show that M is a Noetherian R-module if and only if N and M/N are Noetherian.
   (d) Let R be a Noetherian ring and let M be an R-module of finite type. Show that M is a Noetherian R-module.

4. (a) When is a ring R called a unique factorization domain (UFD)?

(b) Prove that every proper non-zero ideal of a principal ideal domain R is the product of maximal elements in the proper ideals of R (maxp) whose collection is uniquely determined.

(c) If R is a unique factorization domain. Let p be a non-zero element of R which is not a unit. Prove that p is a prime element of R if and only if (p) is a non-zero prime ideal of R.

(a) i. Let R be a ring and $R[X]$ be the polynomial ring over R. When is a polynomial $f \in R[X]$ said to be primitive?

  ii. Let K be the quotient field of R. Prove that for every non-zero polynomial $f \in K[X]$ there is a non-zero $a \in K$ such that $af \in R[X]$ is primitive.

(b) Prove that the product of two primitive polynomial is primitive.

(c) Let R be a unique factorization domain and K be the quotient field of R. Let $f \in R[X]$ be a primitive polynomial of positive degree. Show that f is irreducible in $R[X]$ if and only if f is irreducible in $K[X]$