MTS 105 LECTURE 5: SEQUENCE AND SERIES

1.0 SEQUENCE

A sequence is an endless succession of numbers placed in a certain order so that there is a first number, a second and so on. Consider, for example, the arrangement

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\dots\dots, \frac{1}{n}, \dots..$$

This is a sequence where n^{th} number is attained by taking the reciprocal of n. We can have many more examples, such as

$$1,2,3, \ \ldots\ldots\ldots\ldots, n, \ldots.$$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\dots\dots, \frac{n}{n+1}, \dots.$$

$$-1, 1, -1, 1, \ldots..(-1)^2 \ \ldots\ldots\ldots \text{ Etc}$$

So, in general, a sequence is of the form

$$V_1, V_2, V_3, \ \ldots\ldots\ldots\ldots\ldots \ V_n, \ \ldots..$$

Where V_i's are real numbers and each V_i has a duplicate position. This sequence in the notational form is written as $<V_n>$ or $\{V_n\}$ where V_n is the n^{th} (general) term of the sequence.

Example 1:

1. Given that $U_r = 2_r + 1$, then

$V_1 = 2(1) + 1 = 3$

$V_2 = 2(2) + 1 = 5$

$V_3 = 2(3) + 1 = 7$

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Example 2:

2. Given the sequence $1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \ldots ....$, find the Nth term

Solution: By , we see that the sequence of terms can be written as follows:

$V_1 = 1 = \frac{1}{1^2},$

$V_3 = \frac{1}{9} = \frac{1}{3^2},$

Hence the nth term $V_n = \frac{1}{n^2},$

$V_3 = \frac{1}{9} = \frac{1}{1^2},$

$V_r = \frac{1}{r^2},$

$\therefore \quad V_r = \frac{1}{r^2},$

3. Given the sequence 1,4,9,16,25, ……………….. Find the V_r

$V_1 = 1 = 1^2$

$V_2 = 4 = 2^2$

$V_3 = 9 = 3^2$

$V_4 = 16 = 4^2$

$V_5 = 25 = 5^2$

$\therefore \quad V_r = r^2$

4. Find the first five terms of the sequence defined as follows

$V_1 = 1$, $V_2=3$, $V_r=3V_{r-1} - V_{r-2}$

$V_3 = 3V_2 - V_1 = 3(3) - 1 = 8$

$V_4 = 3V_3 - V_2 = 3(8) - 3 = 21$

$V_5 = 3V_4 - V_3 = 3(21) - 8 = 55$

∴ The first five terms are 1,3,6,21,55

1.1 SERIES

A Series is obtained by forming the sum of the terms of a sequence. A finite series is obtained if a first number of terms of the sequence are summed, otherwise is an infinite series.

The sum of the first n^{th} terms of the sequence V_1, V_2, …..V_n is generally denoted by S_n

$$S_n = V_1 + V_2 + \ldots\ldots\ldots\ldots + V_n = \sum_{r=1}^{n} V_r$$

Here we will be considering basically the following series:

1.1.1 Arithmetic Sequence (or Arithmetic Progression)

An Arithmetic progression (A.P) is a sequence of terms that increases by a constant amount which may be positive or negative. This constant amount is known as the common difference of the series and is usually denoted by d.

The following series are all Arithmetic Progression.

1,2,3,4, ………	Common difference	= +1
$\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \ldots$	Common difference	$= +\frac{1}{4}$
6,3,0,-3 ………..	“ “	= - 3
$-2, -1\frac{1}{2}, -1, -\frac{1}{2}, \ldots$	“ “	$= +\frac{1}{2}$

To ascertain if a given sequence be an A.P, it is necessary to subtract from each term (except the first) the preceeding term. If the results in all cases be the same, the given series will be an A.P, where common difference is this common result.

N^{th} Term of an A.P

First term = a = a + (1 – 1)d

Second term = a + d = a + (2 – 1)d

Third term = a + 2d = a + (3 – 1)d

From the result, it is obviouse that the n^{th} term is a + (n – 1)d.

Arithmetic Mean

To find the arithmetic mean between a and b. Let x be the required A.M. Then a, x, b will be three consecutive terms of an A.P. The common difference of the A.P is (x – a) and also (b – x).

$\therefore \quad x - a = b - x$

i.e $\quad 2x = a + b$

$$x = \frac{a + b}{2}$$

More generally, the n arithmetic means between a and b are the n quantities that, when inserted between a and b, form with them (n + 2) successive times of an A.P.

N Arithmetic Means between a and b

Let d be the common difference of the A.P formed. Then b is the $(n + 2)^{th}$ term of the A.P.

$\therefore \quad b = a + (n + 1)\, d$

$\therefore \quad d(n + 1)\, b - a$

$\therefore \qquad d = \frac{b - a}{(n + 1)}$

The required arithmetic means are a + d, a + 2d, ..., a + nd,

Where $d = \frac{b - a}{(n + 1)}$

The Sum S of n terms of an A.P

To find the sum S of n terms of an A.P whose first term is a and whose common difference is d, we must note that the last term of the A.P will be a + (n – 1)d.

$\therefore \qquad S = a + (a + d) + (a + 2d) + \ldots\ldots\ldots\ldots.. + [a + (n – 1)d] \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad 1$

Reversing the series, we get

$S = [a + (n – 1)d] + [a + (n – 2)d] + [a + (n – 3)d] + \ldots\ldots\ldots + a \ldots\ldots\ldots\ldots. 2$

Hence, (1) + (2) gives:

$2S = [2a + (n – 1)d] + [2a + (n – 1)d] + [2a + (n – 1)d] + \ldots\ldots$ to nterms

$= n\ [2a + (n – 1)d[$

$\therefore \qquad S = \frac{n}{2}\{2a + (n – 1)d\}$

Note: Using l for the last term, {a + (n – 1)d}, the result can be written as

$S = \frac{n}{2}(a + l)$

Examples

1. The sum of three consecutive terms of an A.P is 18, and their products is 120. Find the term

Solution:

Let the term be: a – d, a, a + d (we should take the term a, a+d, a+2d but it is simplier to take a – d, a, a + d)

∴ $a - d + a + a + d = 18$

$3a = 18$

∴ $a = 6$

The Product = (a – d) a (a +) = 6(6 – d)(6 + d) = 120

∴ $6(6^2 - d^2) = 120$

∴ $6(36 - d^2) = 120$

$36 - d^2 = 20$

∴ $d^2 = 36 - 20$

∴ $d = \pm 4$

If d = 4, the terms are 2, 6, 10 and if d = - 4 they are 10, 6, 2. The numbers are the same but form two different A.Ps.

2. The first term of an arithmetic series is 7, the last term is 70, and the sum is 385. Find the number of terms in the series and the common difference.

Solution

Here a = 7, l = 70, Sn = 385

But Sn = $\frac{n}{2}(a + l)$

∴ $385 = \frac{n}{2}(7 + 70) = \frac{77n}{2}$

∴ $77n = 770$

$n = 10$

Since $l = 70$, then $7 + (10 – 1)d = 70$

∴ $7 + 9d = 70$

$9d = 63 \Rightarrow d = 7$

∴ n = 10 is member of terms of the series and the common difference = 7

3. Find the number of terms in an A.P whose first term is 5, common difference 3, and sum 55.

Solution:

Let n be the required number of terms

∴ $55 = \frac{1}{2}n\,(7 + 3n)$

∴ $110 = 7n + 3n^2$ i.e $3n^2 + 7n – 110 = 0$.

∴ $(3n + 22)\,(n – 5) = 0$, ∴ $n = \frac{-22}{3}$ or 5

But n must be a positive integer ∴ $n = 5$

4. The sum of the first n terms of a series is $2n^2 – n$. Find the nth term and show that the series is an A.P.

Solution:

Using n = 1 in the sum for n terms, it is seen that the first term = 2 – 1 = 1.

Using n = 2, the sum of the first two terms = 8 – 2 = 6, therefore the second term is 5

Using n = 3, the sum of the first three terms = 18 – 3 = 15, therefore the third term = 15 – 6 = 9

Replacing n by (n – 1) the sum of the first (n – 1) terms is

$2(n-1)^2 - (n-1) = 2n^2 - 4n + 2 - n + 1$

$= 2n^2 - 5n + 3$

$\therefore$ n^{th} term = sum of first nterms – sum of first (n – 1) term

$= 2n^2 - n - (2n^2 - 5n + 3)$

$= 2n^2 - n - 2n^2 + 5n - 3$

$= 4n - 3$

The series is 1,5,9,…… (4n – 3), which is an A.P of common difference 4.

5. The sum of five numbers in A.P is 25 and the sum of the series be d, then the terms are (a – 2d), (a – d), a, a + d, a + 2d.

From the question

$(a - 2d) + (a - d) + a + (a + d) + (a + 2d) = 25$

$\therefore$ $5a = 25$

$\therefore$ $a = 5$

Also, $(a - 2d)^2 + (a - d)^2 + a^2 + (a + d)^2 + (a + 2d)^2 = 165$

$a^2 - 4ad + 4d^2 + a^2 - 2ad + d^2 + a^2 + (a^2+2ad+d^2) + (a^2+4ad+4d^2) = 165$

$\therefore$ $5a^2 + 10d^2 = 165$

i.e $a^2 + 2d^2 = 33$

$2d^2 = 8$

$\therefore$ $d^2 = 4$

$\therefore \qquad d = \pm 2$

Hence, the series is 1,3,5,7,9 ….

1.2.2 Geometric Sequence (Geometric Progression)

The geometrical progression (G.P) is a sequence of terms that increase or decrease in a constant ratio. This constant ratio is known as the common ratio of the series and is usually denoted by r.

The following examples are all geometrical progressions (G.P)

(i) 2, 4, 8, 16, ………………….. Common ratio $+2$

(ii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, …………………… Common ratio $+\frac{1}{2}$

(iii) $-\frac{1}{3}, 1, -3, 9$, ……………….. Common ratio -3

(iv) $-2, \frac{1}{2}, -\frac{1}{3}, \frac{1}{32}$,……………….. Common ratio $-\frac{1}{4}$

If, in a given sequence, the ratio of each term to the preceding term is the same for all terms, the sequence must be a G.P with this ratio as the common ratio.

n^{th} term of a G.P

First term = $a = a.r^0 = ar^{1-1}$

Second term = $ar = a.r^{2-1}$

Third term = $ar^2 = ar^{3-1}$

From these it can be seen that the n^{th} term is $a.r^{n-1}$.

The geometric mean (G.M) between two quantities a and b is that quantity which when inserted between a and b, forms with them three successive terms of a G.P.

Geometric Mean Between a and b

Let x be the required G.M. Then, a,x,b will be three successive terms of a G.P. The common ratio of the G.P will be x/9 and b/x

$\therefore \quad \frac{x}{9} = \frac{b}{x}$

$\therefore \quad x^2 = ab$

$\therefore \quad x = \pm \sqrt{ab}$

More generally, the n geometric means between a and b are the n quantities that when inserted between a and b form with them (n + 2) terms (successive) of a G.P.

n Geometric means between a and b

Let r be the common ratio of the G.P formed. Then, b is the $(n + 2)^{th}$ term of the G.P.

$\therefore \quad b = ar^{n+1}$

i.e $\quad r^{n+1} = b/a$

$\therefore \quad r = \sqrt[n+1]{b/a}$

Using this value of r, the required geometric means will be ar, ar^2, …………, ar^2

Sum of the first n terms of a G.P

To find the sum of the first n terms of a G.P, whose common ratio is r and first term a.

Let S_n be the required sum. Then,

$$Sn = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1} \ldots\ldots \quad (1)$$

$$\therefore \quad rSn = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n \ldots\ldots \quad (2)$$

(1) – (2) gives, $Sn(1 - r) = a - ar^n$

$= a(1 - r^n)$

$\therefore \qquad Sn = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$

Note:

$$Sn = \begin{cases} \frac{a(1-r^n)}{1-r}, & /r/ < 1 \\ \frac{a(r^n-1)}{r-1}, & /r/ > 1 \end{cases}$$

The value that Sn approaches as $n \longrightarrow \infty$ is known as its sum to infinity ($S\infty$)

From the previous result

$$Sn = a\,\frac{(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Therefore as $n \longrightarrow \infty$ (i.e as n approaches infinity) $r^n \longrightarrow 0$, if $/r/ < 1$, and the second term becomes negligible, whilst r^n n $\longrightarrow \pm\infty$ when $/r/ > 1$ and Sn is then infinite.

Thus, when $/r/ < 1$, $S\infty = \frac{a}{1-r}$

If the sum to infinity is finite, as in this case, the series is said to be convergent.

When $/r/ > 1$, $S\infty = \pm\infty$ and the series is divergent

If r = 1, the series is a + a + a + ….. and so Sn = na

Hence, Sn $\longrightarrow +\infty$ if $a > 0$ but Sn $\longrightarrow -\infty$ if $a < 0$ (both dgt)

If r = - 1, the series is a, - a + a – a + ….. Hence Sn = 0 or a

Example

1. Insert three geometric means between $2\,{}^1/_4$ and ${}^4/_9$

Solution

Let r be the common ratio of the G.P formed. Since ${}^4/_9$ is the fifth term of the G.P

$^4/_9 = {}^9/_4\, r^4$ ∴ $r^4 = {}^{16}/_{81}$

∴ $r^2 = {}^4/_9$ (r real) ∴ $r = \pm\, {}^2/_3$

Therefore required geometric means are $^3/_2$, 1, $^2/_3$ or - $^3/_2$, 1, - $^2/_3$

2. In a geometric progression, the first term is 7, the last term 448 and the sum 889. Find the common ratio.

Solution:

Let r be the common ratio, n the number of terms, and Sn the sum of nterms.

∴ $S_n = 889 = \frac{7(1-r^n)}{1-r}$ ……………… **(1)**

Also $448 = 7r^{n-1}$ …………………………….. **(2)**

Using (2) in (1), $889 = \frac{7-7r^n}{1-r} = \frac{7-448r}{1-r}$

$889(1 - r) = 7 - 448r$

∴ $889 - 889r = 7 - 448r$

∴ $882 = 441r$

⟹ $r = 2$

3. Find the sum of the first six terms of the geometric series whose thrid term is 27 and whose sixth term is 8.

Find how many terms of this series must be taken if their sum is to be within $^1/_{10}$% of the sum to infinity.

Solution:

Let r be the common ratio of the series and a the first term.

∴ $ar^2 = 27$ …………………………….. (1)

$ar^5 = 8$ ……………………………… (2)

(2) ÷ (1) gives $r^3 = {}^8/_{27}$ $\therefore$ $r = \frac{2}{3}$

Using this in (1), $9\ x\ \frac{4}{9} = 27$ $\therefore$ $a = \frac{243}{4}$

The sum of the first six terms of this series

$$= \frac{9(1-r^n)}{1-r} = \frac{243}{4}\ \frac{\{1-\left(\frac{2}{3}\right)^6\}}{1-\frac{2}{3}} = \frac{243}{4}\ \frac{\{1-\frac{26}{36}\}}{\frac{1}{3}}$$

$$= \frac{729}{4}\ \left\{1 - \frac{26}{36}\right\} = \frac{729}{4} - \frac{729}{4}\ x\ \frac{26}{36}$$

$$= \frac{729}{4} - 16 = \frac{665}{4}$$

Let n be the number of terms required so that their sum shall be ${}^1/_{10}$% less than the sum to infinity. $S\infty$.

Now $S\infty = \frac{9}{1-r}$, and Sn $= \frac{a(1-r^n)}{1-r} = \frac{9}{1-r} - \frac{ar^n}{1-r}$

$\therefore$ $S\infty - Sn = \frac{ar^n}{1-r}$

But $S\infty$ - Sn $= \frac{1}{10}$ percent of $S\infty$

$\therefore$ $\frac{S\infty - Sn}{S\infty} = \frac{1}{1000}$

$\therefore$ $\frac{ar^n}{1-r} \div \frac{a}{1-r} = \frac{1}{1000}$ i.e. $r^n = \frac{1}{1000}$

(i.e.) $\left(\frac{2}{3}\right)^n = \frac{1}{1000}$

Taking logs to the base of 10, we get

n[log2 – log3] = - 3

$\therefore$ $n = \frac{3}{log_{10}3 - log_{10}2} = \frac{3}{0.47712-0.30103} = \frac{3}{0.17609} = 17.03$

$\Longrightarrow$ Regd no is 18 (i.e. next +ve integer > 17)

The powers of the First n Natural Numbers

Note: The method adopted in the summation of the second and third powers of the first n natural numbers is the method of undetermined coefficients, and consisted of equating identically the given series to $A + B_n + C_n^2 + \ldots\ldots\ldots$ and then determining the values of the unknown constants A, B, C etc. by the properties of identities.

a. The first n natural numbers form an A.P whose sum S_1 has been shown to be n(n+1)/2.

b. Let the sum of the squares of the first n natural numbers be S_2 and

$$1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A + B_n + C_n^2 + D_n^3 + \ldots\ldots\ldots\ldots \quad (1)$$

Replacing n by (n + 1) in this identity

$$1^2 + 2^2 + 3^2 + \ldots + (n + 1)^2 \equiv A + B((n + 1) + C(n + 1)^2 + D(n + 1)^3 + \ldots \quad (2)$$

(2) – (1) gives:

$$(n + 1)^2 \equiv B + C(2n+1) + D(3n^2+3n+1) + \ldots\ldots\ldots\ldots\ldots \quad (3)$$

In the identity (3) the singular power of n on the left-hand side (L.H.S) is the second, and therefore the highest power of n on the right-hand side (R.H.S) must be the second and all other coefficients after D must varies.

Equating coefficient of n in (3)

n^2 : $1 = 3D$ $\quad\therefore\ D = \frac{1}{3}$

n : $2 = 3D + 2C = 1 + 2C$ $\quad\therefore\ C = \frac{1}{2}$

Unity: $1 = B + C + D = B + \frac{1}{2} + \frac{1}{3}$ $\quad\therefore\ B = \frac{1}{6}$

$\therefore\quad 1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

Using n = 1, $1^2 = A + \frac{1}{6} + \frac{1}{2} + \frac{1}{3}$

$\therefore$ $A = 0$

Thus, $1^2 + 2^2 + 3^2 + \ldots + n^2 = +\frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

$$= \frac{n}{6}(1 + 3n + 2n^2)$$

$$= \frac{n\,(n+1)(2n+1)}{6}$$

Alternative Method

It is known that, for all values of n,

$(n + 1)^3 - n^3 \equiv 3n^2 + 3n + 1$ ……………………… (1)

Replacing n by (n-1), (n-2), ……….., 2, 1 in succession,

$n^3 - (n - 1)^3 \equiv 3(n - 1)^2 + 3(n - 1) + 1$ …………. (2)

$(n - 1)^3 - (n - 2)^3 \equiv 3(n - 2)^2 + 3(n - 2) + 1$ ……. (3)

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$3^3 - 2^3 \equiv 3.2^3 + 3.2 + 1$ …………………………. (n-1)

$2^3 - 1^3 \equiv 3.1^3 + 3.1 + 1$ …………………………. (n)

Adding these n identities

$(n + 1)^3 - 1^3 \equiv 3(1^2 + 2^2 + \ldots + n^2)$

$+ 3(1 + 2 + 3 + \ldots + n) + 2$

$\equiv 3S_2 + \frac{3}{2}n(n+1) + n$

$\therefore$ $3S_2 \equiv (n + 1)^3 - 1^3 - 3\frac{n}{2}(n + 1) - n$

$= n^3 + 3n^2 + 3n - 3\frac{n}{2}(n + 1) - n$

$= \frac{n}{2}\{2n^2 + 6n + 6 - 3n - 3 - 2\}$

$= \frac{n}{2}\{2n^2 + 3n + 1\} = \frac{n}{2}(n + 1)(2n + 1),$

$\therefore \quad S_2 = \frac{n(n+1)(2n+1)}{6}$

(c) Let S_3 be the sum of the cubes of the first n natural numbers and

$$1^3 + 2^3 + 3^3 + \ldots + n^3 \equiv A + Bn + Cn^2 + Dn^3 + En^4 \ldots\ldots \quad (1)$$

Replacing n by (n+1) in this identity,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1) \equiv A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \ldots \quad (2)$$

(2) – (1) gives

$$(n+1)^3 \equiv B + C(2n+1) + D(3n^2 + 3n + 1) + E(4n^3 + 6n^2 + 4n + 1) + \ldots\ldots \quad (3)$$

The highest power of n on the L.H.S of (3) is the third, and therefore all coefficients after E on the R.H.S of (3) must vanish.

Equating coefficients in identity (3),

$n^3 \quad : \quad 1 = 4E \quad \therefore \quad E = {}^{1}/_{4}$

$n^2 \quad : \quad 3 = 6E + 3D = \frac{3}{2} + 3D \quad \therefore \quad 3D = \frac{3}{2} \quad \therefore \quad D = \frac{1}{2}$

$n \quad : \quad 3 = 4E + 3D + 2C = 1 = 1 + \frac{3}{2} + 2C \quad \therefore \quad \frac{1}{2} = 2C, \quad \therefore \quad C = \frac{1}{4}$

Unity: $1 = B + C + D + E = B + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, \quad \therefore \quad B = 0$

Hence, $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 \equiv A + \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4.$

Using n =1 in this,

$1^3 = A + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$, $\therefore$ $A = 0$,

$\therefore$ $S_3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4 \equiv \frac{n^2}{4}(1 + 2n + n^2) \equiv \left\{\frac{n}{2}(n + 1)\right\}^2 = S_1^2$

As in (b), the alternative method consists of using

$(n+1)^4 - n^4 \equiv 4n^3 + 6n + 4n + 1$

and replacing n by (n – 1), (n – 2), ……………, etc, in succession and then introducing the values of S_1 and S_2 already determined.

Example: Find the sum of the series

(i) $1^2 - 2^2 + 3^2 - 4^2 + \ldots\ldots\ldots\ldots\ldots$ to 2n terms

(ii) $1.3^2 + 2.4^2 + 3.5^2 + \ldots\ldots\ldots\ldots\ldots$ to n terms

(iii) $1^3 + 3^3 + 5^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ to n terms

Solution

(i) Let the required sum the S_2n

$\therefore$ $S_2n = (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + \text{ to 2n terms}) - 2(2^2+4^2+6^2+\ldots\ldots\ldots+ \text{ to nterms})$

$= \sum_{r=1}^{r=2n} r^2 - 8(1^2 + 2^2 + 3^2 + \cdots + to\ n\ term)$

$= \sum_{r=1}^{2n} r^2 - 8\sum_{r=1}^{n} r^2$

$= \frac{2n(2n+1)(4n+!)}{6} - \frac{8n(n+1)(2n+1)}{6}$ {Using formular fo S_2}

$= \frac{n(2n+1)}{3}\{4(n + 1) - 4(n + 1)\} = \frac{-(2n+1).3}{3}$

$= - n(2n + 1)$

(ii) The r^{th} term of the series $= r(r+2)^2 = r^3 + 4r^2 + 4r$, therefore sum of nterms of the series

$= \sum_{r=1}^{n}(r^3 + 4r^2 + 4r)$

$=\quad \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r$

$=\quad \left\{\frac{n(n+1)}{2}\right\}^3 - \frac{4n(n+1)(2n+1)}{6} + 4 \times \frac{n}{2}(n + 1)$

$=\quad \frac{n}{12}(n + 1)\{3n(n + 1) + 8(2n + 1) + 24\}$

$=\quad \frac{n(n+1)}{12}\{3n^2 + 19n + 32\}.$

(iii) The r^{th} term of the series is $(2r - 1)^3 \equiv 8r^3 - 12r^2 + 6r - 1$, therefore sum of n terms

$=\quad \sum_{r=1}^{n}(8r^3 + 12r^2 + 6r - 1)$

$=\quad 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - n)$

$=\quad 8\left\{\frac{n(n+1)}{2}\right\}^2 - 12\,\frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} - n$

$=\quad 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$

$=\quad n\{(2n^3+4n^2+2n) - (4n^2+6n+2) + (3n+3) - 1\}$

$=\quad n\{2n^3 - n\} = n^2(2n - 1).$

Note: Example (i) can be solved readily as follows:

$1^2 - 2^2 + 3^2 - 4^2 + \dots\dots\dots\dots + (2n - 1)^2 - (2n)^2$

$=\quad (1^2 - 2^2) + (3^2 - 4^2) + \dots\dots\dots + \{(2n - 1)^2 - (2n)^2\}$

$=\quad - 1.3 - 1.7 - 1.11 - \dots\dots\dots\dots - 1(4n - 1)$

$=\quad -\frac{1}{2}\,n\,(4n + 2) = -n(2n + 1)$