UNIVERSITY OF AGRICULTURE, ABEOKUTA

DEPARTMENT OF PHYSICS

PHS 472 (Mathematical Physics) (3 units)

References:

1. Morse and Feshback , Methods of Mathematical Physics
2. Jeffreys and Jeffreys, Methods of Mathematical Physics
3. Courant and Hilbert, Methods of Mathematical Physics
4. Eugene Butkov, Mathematical Physics
5. Stephenson: Mathematical Methods for the Physics and Engineering
6. Riley: Mathematical Methods for the Physical Sciences

Lecture 1(complex numbers)

1.1 ***Preamble***: Classical Physics,with a few exceptions, relies on real numbers for its mathematical basis. Quantum mechanics marked the entry of complex numbers,in a fundamental way,into physics.
Here in this lecture,we define what is a complex number and we review the main properties of complex numbers for use in the remainder of this course.

1.2 ***Definition of complex number*** **:**A complex number z is an ordered pair (a,b) of real numbers a and b,written as $z = a + ib$, where a,b are real numbers and i,called the imaginary unit,has the property that $i^2 = -1$.

1.3 ***Operations with complex numbers*****:**

(a) *in cartesian(or rectangular)coordinates representation*

Addition and Subtraction of complex numbers are easy;just as for 2-D vectors,the real and imaginary parts are added or subtracted separately:

$$(a+bi)+(c+di)=(a+c)+(b+d)i \qquad (1.31)$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i \qquad (1.32)$$

Multiplication and division are more subtle.

$$(a+bi)(c+di)=(ac-bd)+(bc+ad)i \qquad (1.33)$$

$$\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2} \qquad (1.34)$$

(b) *in polar representation*

$$\begin{aligned} e^{i\theta} &= 1+(i\theta)+\frac{1}{2}(i\theta)^2+\frac{1}{6}(i\theta)^3+\frac{1}{24}(i\theta)^4+\frac{1}{120}(i\theta)^5+\ldots \\ &= \left(1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4-\ldots\right)+i\left(\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\ldots\right) \\ &= \cos\theta+i\sin\theta \qquad (1.35) \end{aligned}$$

(c) *Powers and Roots*

Consider $z = a + bi = \mathrm{R}e^{i\theta} = R\cos\theta + iR\sin\theta$ (1.36)

R is called the **modulus** or the absolute value of z, and θ is called the **argument** of z. Note that θ must be expressed in radians.

Clearly, the n-th **power** of z is given by:

$$z^n = (a+bi)^n = (\mathrm{R}e^{i\theta})^n = R^n e^{in\theta} \qquad (1.37)$$

Where by **de Moivre's** theorem , $e^{in\theta} = \cos n\theta + i\sin n\theta$ (1.38)

Similarly, the n-th **root** of z is given by:

$$\sqrt[n]{z} = z^{1/n} = (a+bi)^{1/n} = (\mathrm{R}e^{i\theta})^{1/n}$$

$$= R^{1/n}\left(e^{i\left(\frac{\theta+2k\pi}{n}\right)}\right) = R^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

$$k = 0,1,2,\ldots.n-1 \qquad (1.39)$$

1.4 Tutorial 1

1. Express each of the following operations as a complex number :

(a) $(3-2i)^3$, (b) $\left(\frac{1-i}{1+i}\right)^{10}$, (c) $\frac{(1+i)(2+3i)(4-2i)}{(1+2i)^2(1-i)}$, (d) $\frac{5}{3-4i} + \frac{10}{4+3i}$

2. Express in polar form: (a) $-2-2i$, (b) $3\sqrt{3}+3i$

3. Express in Cartesian form : (a) $[2(\cos 25^0 + i\sin 25^0)][5(\cos 110^0 + i\sin 110^0)]$,

(b) $\frac{12cis16^0}{(3cis44^0)(2cis62^0)}$

4. Express the function $f(z) = \ln z$ in both (i) Cartesian and (ii) plane-polar coordinates.

5. Obtain the 4 complex numbers, whose 4th power is 1+ i

Lecture 2(Analytic function of complex variables)

2.1 Definition of function of complex variables: A function f(z) of complex variables is given by $f(z) = U(x, y) + iV(x, y)$;
where U and V are complex variables.
For example, given $f(z) = z^2 + 3z$, it can be shown that:
$U = x^2 - y^2 + 3x$; and $V = 2xy + 3y$

2.2 Definition of Analytic function: A function *f(z)* is analytic in a domain D if *f(z)* is
(i) defined and (ii) differentiable at all points in D.
For example, given complex constants $c_0, c_1, c_2, ...c_n$,
the polynomials $f(z) = c_0 + c_1 z + c_2 z^2 + ...+ c_n z^n$ are analytic in the entire complex plane.

2.3 Cauchy-Riemann Equations (a test for analyticity of a Complex function)
Given **,** $f(z) = U(x, y) + iV(x, y)$, *f(z)* is analytic in a domain D iff :

$$U_x = V_y$$

$$U_y = -V_x \qquad (2.31)$$

For example, Consider $f(z) = z^2$
Clearly, it can be verified that:
$U_x = 2x$, $V_y = 2x$, $\Rightarrow U_x = V_y$
and $U_y = -2y$, $V_x = 2y$, $\Rightarrow U_y = -V_x$
which therefore $\Rightarrow f$ is analytic.

2.4 Cauchy Integration Formula (a consequence of analyticity of a Complex function)
Given C is a simple closed curvature in a domain D and let a be an interior point to C; then

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz \qquad (2.41)$$

where the contour C is taken in the positive sense.
Note that : (i) $f(z)$ is analytic at the point a ,
(ii) its derivatives of all orders are also analytical at the point a
In other words,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{z-a}dz \qquad (2.42)$$

Where n is the order of the derivative.

2.5 Tutorial 2

1. If $f(z) = z^3$, show that $f(z)$ is analytic.

2. Use **Cauchy integral formula** to evaluate the following integrals :

(i) $\oint_c \frac{z}{2z+1} dz$, (ii) $\oint_c \frac{\sin z}{(z-\frac{\pi}{4})^2} dz$ where c is the circle $|z| = 2$

Lecture 3(Power Series of a complex function)

3.1 Definition of Power Series of a complex function: A power series expansion or development of the function f(z) of complex variables is given by

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \;=c_0+c_1(z-a)+c_2(z-a)^2+.....+c_n(z-a)^n+.... \quad (3.1)$$

i.e. *f(z)* is an infinite series of the form given in equation (3.1) where $a, c_0, c_1, c_2, ...c_n, ...$ are given complex numbers and z is a complex variable about a.

For example, $f(z)=\dfrac{1}{1-z}=1+z+z^2+z^3+.....$

3.2 Taylor series : is a power series of the form given in equation (3.1) where

$$c_n = {f^n(a)}/{n!} \quad (3.2)$$

i.e. $$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+...... \quad (3.3)$$

3.3 Tutorial 3

1. Expand the following function in Taylor's series: $f(z)=\dfrac{1}{1-z}$ around z $= i$

Lecture 4(Poles and Residue)

4.1 Singular Point: A singular point of a function *f(z)* is a value at which *f(z)* fails to be analytic.

If *f(z)* is analytic everywhere in some region except at an interior point $z=a$, then $z=a$ is called an isolated singularity of *f(z)***.**

For example, if $f(z)=\frac{1}{(z-3)^2}$, then, $z=3$ is an isolated singularity of $f(z)$

4.2 Poles: Consider the function ; $f(z)=\frac{\phi(z)}{(z-a)^n}, \phi(a)\neq 0$ (4.21)

$f(z)$ has an isolated singularity at $z=a$ which is called a **pole** of order n.

If $n=1$, the pole is often called a simple pole; if $n=2$, it is called a double pole,etc.

For example, consider $f(z)=\frac{z}{(z-3)^2(z+1)}$

$f(z)$ has a pole of order 2(or double pole) at $z=3$, and a pole of order 1(simple pole) at $z = -1$.

4.3 Laurent's Series: This is an extension of Taylor's series. Here *f(z)* is given as :

$$f(z)=\left\{\frac{a_{-n}}{(z-a)^n}+\frac{a_{-n+1}}{(z-a)^{n-1}}+.....+\frac{a_{-1}}{(z-a)}\right\}+\left\{a_0+a_1(z-a)+a_2(z-a)^2+......\right\} \quad (4.31)$$

{ Principal part } { analytical part }

4.4 Residue : The coefficient a_{-1} in equation (4.31) is called the **residue** of *f(z)* at the pole $z=a$.

It is of considerable importance and can be found from the formula in equation (4.41) :

$$a_{-1}=\lim_{z\to a}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n f(z)\right\} \quad (4.41)$$

where n is the order of the pole.

4.5 Residue Theorem: This is given by equation (4.51) as:

$$\oint_c \frac{dz}{(z-a)^n}=\begin{cases}0, n\neq 1\\ 2\pi i, n=1\end{cases} \quad (4.51)$$

$$\Rightarrow \oint_c f(z)dz = 2\pi i a_{-1}$$

4.6 Tutorial 4

1. Find the **residues** at those **singular points** which lie inside the circle $|z| = 2$

(i) $\dfrac{3z+6}{(z+1)(z^2+16)}$, (ii) $\dfrac{z^4}{z^2 - iz + 2}$

2. Using **residue theorem**, evaluate $\oint_c \dfrac{5z^2 - 3z + 2}{(z-1)^3}\,dz$, where c is the unit circle.

3.(a) Express $f(z) = \dfrac{z^3}{(z+2)^2}$ as a **Laurent series** about the point $z = -2$;

(b) hence, or otherwise evaluate $\oint_c \dfrac{z^3}{(z+2)^2}\,dz$ where c is the circle $|z| = 2$

Lecture 5(Differential Equations)

5.1 Definition of Differential Equation : A differential equation is an equation which involves at least 1 derivative of an unknown function.

Examples are: $\frac{dy}{dx} = \sin x$ (5.11)

$x\frac{dy}{dx} = y^2 + 1$ (5.12)

Many problems in Physics,chemistry,engineering,etc can be formulated in the form of differential Equations. Thus differential equations play an important role in the application of mathematics to Scientific problems.

5.2 Illustrative examples of differential equations

(1) Rate of decay of a radioactive substance is proportional to the amount present.

i.e. $\frac{dy}{dt} = ky$ (5.21)

where y is the amount of the radioactive substance present at time t and k is a constant.

(2) Newton's Law of cooling states that the rate of change of temperature in a cooling body is proportional to the difference in temperature between the body and its surroundings.

i.e. $\frac{d\theta}{dt} = k(\theta - \theta_R)$ (5.22)

where θ_R is temperature of the surrounding and k is a constant.

(3) Newton's Law of gravitation states that the acceleration of a particle is inversely proportional to the square of the distance between the particle and the centre of the earth.

i.e. $\frac{d^2x}{dt^2} = \frac{k}{x^2}$ (5.23)

5.3 Basic Concepts of Partial Differential Equation(P.D.E.)

(1) Definition **:** A p.d.e. is any equation of the type:

$F(x, y, z, \ldots u, u_x, u_y, \ldots u_{xx}, u_{xy}, \ldots\ldots) = 0$ (5.31)

Which involves ***several independent variables*** x,y,…….***one dependent variable*** u, and ***some*** of its p.d. $u_x = \frac{\partial u}{\partial x}, \ldots \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}$

(2) Order is the order of the highest derivative in the equation.

Consider the following examples :

(i) $\frac{\partial^2 u}{\partial x \partial y} = 2x - y$ is a p.d.e of order 2

(ii) $y\left(\frac{\partial u}{\partial x}\right)^2 = \sin y$ is a p.d.e. of order 1

(3) Linear **:**A p.d.e. is linear if it is of 1st degree in the dependent variable and its partial derivative.

An example $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ is linear, and of 2nd order .

(4) Homogeneous**:** each term of a p.d.e. contains either the dependent variable or one of its
Derivatives ; otherwise nonhomogeneous.

Consider the example $a_1\frac{\partial^2 y}{\partial x^2} + a_2\frac{\partial y}{\partial x} + a_3 y = f(x)$ (5.32)

Where a_1, a_2, a_3 are real constants, and $a_1 \neq 0$

If $f(x) = 0$, the equation (5.32) is said to be homogeneous.

5.4 Tutorial 5

1. Determine whether each of the following partial differential equations is ***linear*** or ***non-linear***.
State the order of each equation and name the ***dependent*** and ***independent*** variables

(i) $\frac{\partial \varphi}{\partial t} = 4\frac{\partial^2 \varphi}{\partial x^2}$ (ii) $v\frac{\partial^2 v}{\partial r^2} = rst$ (iii) $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 1$

(iv) $x^2\frac{\partial^2 R}{\partial y^2} = y^3\frac{\partial^2 R}{\partial x^2}$ (v) $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$

2. For each of the following partial differential equations state :
(a) the ***dependent*** variable(s);
(b) the ***independent*** variable(s);
(c) the ***order*** of the equation;
(d) the ***degree*** of the equation;
(e) whether the equation is ***linear*** or ***non-linear*****;**
(f) whether the equation is ***homogeneous*** or ***non-homogeneous***.

(i) $\frac{\partial z}{\partial r} + \frac{\partial z}{\partial s} = \frac{1}{z^2}$, (ii) $\psi\frac{\partial \psi}{\partial x} = \frac{\partial^3 \psi}{\partial y^3}$, (iii) $\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = x^2$,

(iv) $\frac{\partial^2 \phi}{\partial x^2} + 2\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2} = 0$, (v) $(x^2 + y^2)\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$

Lecture 6(Classification of Partial Differential Equations)

An equation of the form : $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$ (6.1)
is said to be :

Elliptic if : $B^2 - 4AC < 0$ (6.2)

Parabolic if: $B^2 - 4AC = 0$ (6.3)

Hyperbolic if: $B^2 - 4AC > 0$ (6.4)

Note that A,B,C may be functions of x and y and the type of equation (6.1) may be different in different parts of the xy-plane.
For example, consider equation (6.5) :

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (6.5)$$

Clearly, $A = 1, B = 0, C = 1$

Thus, $B^2 - 4AC = -4$

$\Rightarrow$ equation (6.5) is ***elliptic***

Tutorial 6

1. Classify each of the following equations as ***elliptic***,***hyperbolic*** *or* ***parabolic***.

(i) $\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0$ (ii) $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} = 4$ (iii) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + 3y$

(iv) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$

(v) $(x^2 - 1)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + (y^2 - 1)\frac{\partial^2 u}{\partial y^2} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Lecture 7(Important Linear Partial Differential Equations of the 2nd Order)

7.1 Wave Equation : This is of the form given in equation (7.11) :

In 1-D , $$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} \qquad (7.11)$$

In 3-D, $$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial r^2} = c^2\nabla^2\phi \qquad (7.12)$$

Where $\phi = \phi(r,t)$ and $\nabla^2 = \frac{\partial^2}{\partial r^2} = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right)$

Note that ϕ may be a scalar or a vector as it occurs in ***electromagnetic waves .***

7.2 Helmholtz equation : Consider the wave equation (7.12).
If the time dependence of $\phi = \phi(r,t)$ is of the form :

$$\phi(\bar{r},t) = U(\bar{r})e^{\pm i\omega t} \qquad (7.21)$$

Then, equation (7.12) reduces to :

$$(\nabla^2 + k^2)U(\bar{r}) = 0 \qquad (7.22)$$

Where $$k^2 = \frac{\omega^2}{c^2} \qquad (7.23)$$

An example is the ***time-independent Schrodinger equation*** in Quantum Mechanics.

7.3 Heat Equation or Diffusion Equation : This is of the form given in equation(7.31) as :

In 1-D, $$\frac{\partial\phi}{\partial t} = c^2\frac{\partial^2\phi}{\partial x^2} \qquad (7.31)$$

In 3-D, $$\frac{\partial\phi}{\partial t} = c^2\frac{\partial^2\phi}{\partial r^2} = c^2\nabla^2\phi \qquad (7.32)$$

Where $$\phi = \phi(\bar{r},t) \qquad (7.33)$$

7.4 Laplace Equation : This is of the form given in equation (7.41) as:

$$\nabla^2\phi = 0 \qquad (7.41)$$

Note that $\phi = \phi(\bar{r})$ is the potential equation.

ϕ can represent the following potentials:

(i) ***electric potential*** at any point where there is ***no charge***
(ii) ***gravitational potential*** at a point where there is ***no mass*** present
(iii) ***the temperature*** $T(\bar{r},t)$ in the ***steady-state***,inside a conductor where there is no source or sink of Heat.

7.5 Poisson Equation : This is of the form given in equation (7.51) as:

$$\nabla^2\phi = f(\bar{r}) \neq 0 \qquad (7.51)$$

7.6 Tutorial 7

1 Show that: (i) the **Heat equation** is parabolic. (ii) the **wave equation** is Hyperbolic, (iii) the **Laplace equation** is elliptic, (iv) the **Triconi equation** : $y\phi_{xx} + \phi_{yy} = 0$ is of mixed type(elliptic in the upper half-plane and hyperbolic in the lower half-plane).

Lecture 8(Solution of Partial Differential Equations)

8.1 Definition of solution : A function $\phi(x, y, ...)$ is said to be a solution of a Partial Differential Equation if when substituted into the p.d.e., it yields an identity in the independent variable . That is, it satisfies the equation identically.

8.2 Types of solution :

(a) ***general solution*** : one which contains a number of arbitrary independent functions equal to the order of the equation.

(b) ***particular solution*** : one which can be obtained from the general solution by particular choice of the arbitrary function.

(c) ***singular solution*** :is one which cannot be obtained from the general solution by particular choice of the arbitrary function.

8.3 Tutorial 8

1. Stating the arbitrariness thereof, solve (i) $\psi_{xx} = 0$; (ii) $\psi_x = 2xy\psi$

2. The function $\psi(x, y)$ obeys the Laplace equation,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

Show whether or not $\psi = x^3 - 3xy^2$ is a solution.

Lectures 9-12(Second order Partial Differential Equations)

Solution of Problems on ***Second order Partial Differential Equations, Boundary value*** using the ***method of separation of variables.***

Tutorial 9

1. In Quantum mechanics,the time-dependent Schrodinger equation is given by:

$i\hbar\frac{\partial\psi(r,t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(r,t)+V(r)\psi(r,t)$ where $\psi(r,t)$ is a function in Space r and time t.

(i) separate this into space and time parts.

(ii) Deduce that for a free particle(V=0), the spatial part reduces to Helmholtz equation.

(iii) Solve this equation(Helmholtz)by a method of separation of variables in Cartesian coordinates.