COURSE CODE: *MCE 315*

COURSE TITLE: *Theory of Elasticity*

NUMBER OF UNITS: *2 Units*

COURSE DURATION: *Two hours per week*

COURSE DETAILS:

Course Coordinator: **Dr. Engr. Olokode, O.S.** Ph D
Email: olokodeos@unaab.edu.ng
Office Location: **Room 3 PG School**
Other Lecturers: **None**

COURSE CONTENT:

Of Theory of Elasticity to Two- and Three-dimensional Problems in Engineering; Stress Concentration round holes; Discs, Wedges under point loading etc. experimental stress analysis, Strain gauging, photo-elasticity and Holography. Approximate methods; Finite element method.

COURSE REQUIREMENTS:

This is a compulsory course for all Mechanical Engineering students in the University. In view of this, students are expected to participate in all the course activities and have minimum of 75% attendance to be able to write the final examination

READING LIST:

Theory of Elasticity (McGraw-Hill Classic Textbook Reissue Series)
Theory of Elastic Stability [Paperback]
Stephen P. Timoshenko (Author), James M. Gere (
The Theory of Plates and Shells (McGraw-Hill Classic Textbook Reissue Series) [Paperback]
S. Timoshenko
An Introduction to the Theory of Elasticity [Paperback]
R. J. Atkin (Author), N. Fox

LECTURE NOTES

Introduction to Finite Element Methods

(MCE 315) THEORY OF ELASTICITY

BY
DR. OLOKODE O.S

MECHANICAL ENGINEERING DEPARTMENT

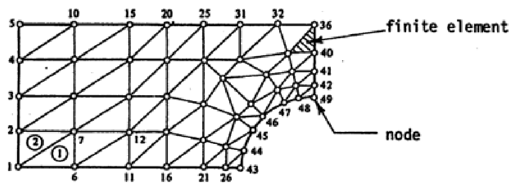
UNIVERSITY OF AGRICULTURE, ABEOKUTA

Need for Computational Methods

- **Solutions Using Either Strength of Materials or Theory of Elasticity Are Normally Accomplished for Regions and Loadings With Relatively Simple Geometry**
- **Many Applications Involve Cases with Complex Shape, Boundary Conditions and Material Behavior**
- **Therefore a Gap Exists Between What Is Needed in Applications and What Can Be Solved by Analytical Closed-form Methods**
- **This Has Lead to the Development of Several Numerical/Computational Schemes Including: Finite Difference, Finite Element and Boundary Element Methods**

Introduction to Finite Element Analysis

The finite element method is a computational scheme to solve field problems in engineering and science. The technique has very wide application, and has been used on problems involving *stress analysis, fluid mechanics, heat transfer, diffusion, vibrations, electrical and magnetic fields*, etc. The fundamental concept involves dividing the body under study into a finite number of pieces (subdomains) called *elements* (see Figure). Particular assumptions are then made on the variation of the unknown dependent variable(s) across each element using so-called *interpolation or approximation functions*. This approximated variation is quantified in terms of solution values at special element locations called *nodes*. Through this discretization process, the method sets up an algebraic system of equations for unknown nodal values which approximate the continuous solution. Because element size, shape and approximating scheme can be varied to suit the problem, the method can accurately simulate solutions to problems of complex geometry and loading and thus this technique has become a very useful and practical tool.



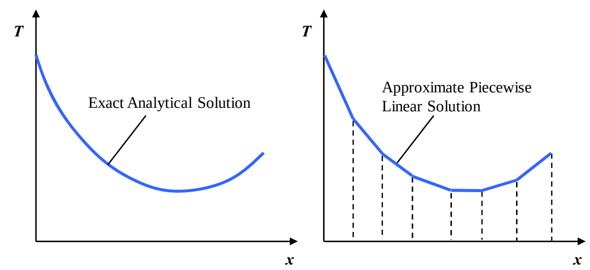
Advantages of Finite Element Analysis

- **Models Bodies of Complex Shape**
- **Can Handle General Loading/Boundary Conditions**
- **Models Bodies Composed of Composite and Multiphase Materials**
- **Model is Easily Refined for Improved Accuracy by Varying Element Size and Type (Approximation Scheme)**
- **Time Dependent and Dynamic Effects Can Be Included**
- **Can Handle a Variety Nonlinear Effects Including Material Behavior, Large Deformations, Boundary Conditions, Etc.**

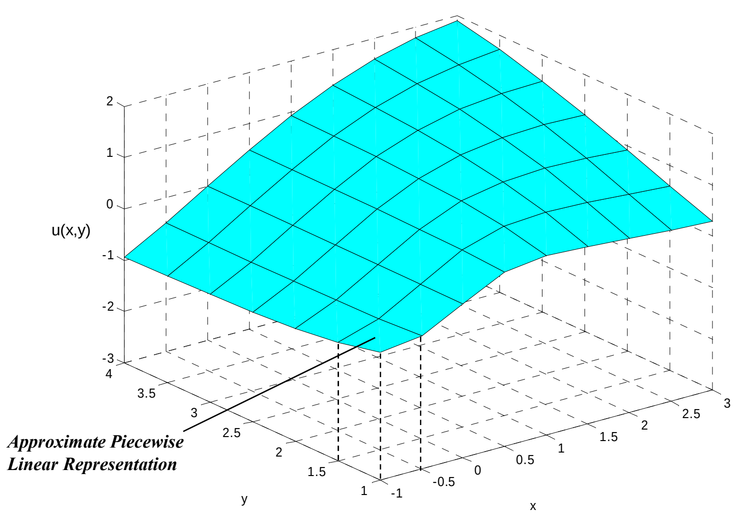
Basic Concept of the Finite Element Method

Any continuous solution field such as stress, displacement, temperature, pressure, etc. can be approximated by a discrete model composed of a set of piecewise continuous functions defined over a finite number of subdomains.

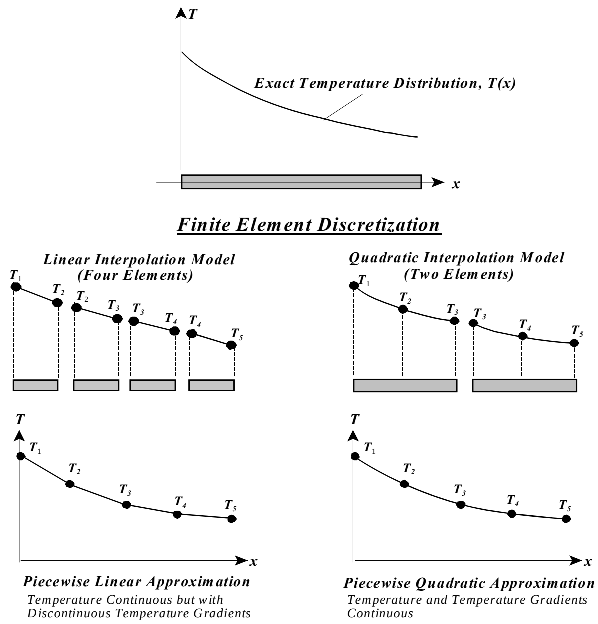
One-Dimensional Temperature Distribution



Two-Dimensional Discretization



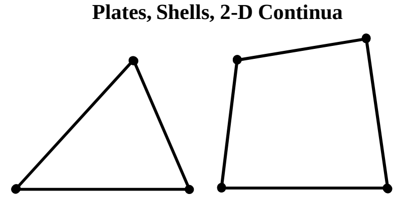
Discretization Concepts



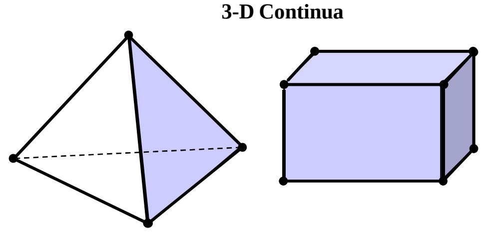
Common Types of Elements

One-Dimensional Elements
Line
Rods, Beams, Trusses, Frames

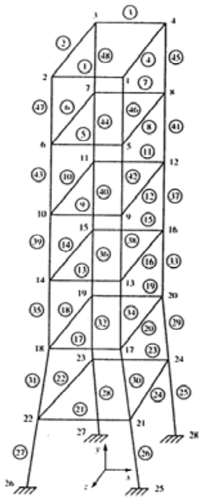
Two-Dimensional Elements
Triangular, Quadrilateral
Plates, Shells, 2-D Continua



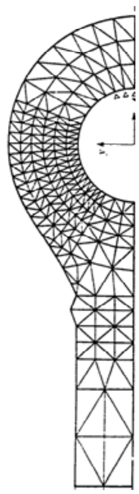
Three-Dimensional Elements
Tetrahedral, Rectangular Prism (Brick)
3-D Continua



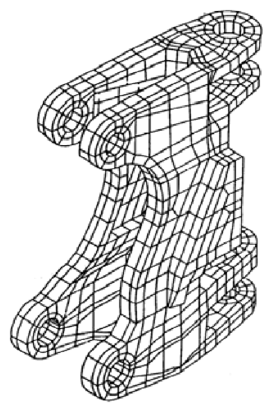
Discretization Examples



One-Dimensional Frame Elements



Two-Dimensional Triangular Elements



Three-Dimensional Brick Elements

Basic Steps in the Finite Element Method Time Independent Problems

- **Domain Discretization**
- **Select Element Type (Shape and Approximation)**
- **Derive Element Equations (Variational and Energy Methods)**
- **Assemble Element Equations to Form Global System**

$$[K]\{U\} = \{F\}$$

[K] = Stiffness or Property Matrix
{U} = Nodal Displacement Vector
{F} = Nodal Force Vector

- **Incorporate Boundary and Initial Conditions**
- **Solve Assembled System of Equations for Unknown Nodal Displacements and Secondary Unknowns of Stress and Strain Values**

Common Sources of Error in FEA

- **Domain Approximation**
- **Element Interpolation/Approximation**
- **Numerical Integration Errors**
  (Including Spatial and Time Integration)
- **Computer Errors (Round-Off, Etc.,)**

Measures of Accuracy in FEA

Accuracy

Error = |(Exact Solution)-(FEM Solution)|

Convergence

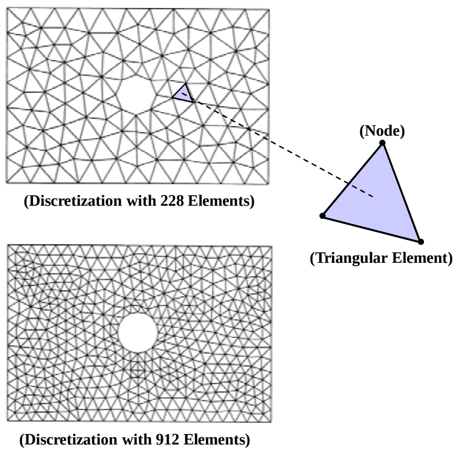
Limit of Error as:

Number of Elements (*h-convergence*)
or
Approximation Order (*p-convergence*)

Increases

Ideally, Error $\rightarrow$ 0 as Number of Elements or Approximation Order $\rightarrow \infty$

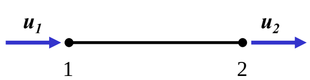
Two-Dimensional Discretization Refinement



One Dimensional Examples
Static Case

Bar Element

Uniaxial Deformation of Bars
Using Strength of Materials Theory



Differential Equation :

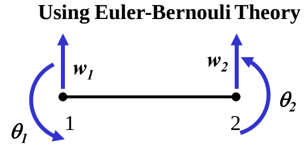
$$-\frac{d}{dx}(au) + cu - q = 0$$

Boundary Condtions Specification :

$$u \ , \ a\frac{du}{dx}$$

Beam Element

Deflection of Elastic Beams
Using Euler-Bernouli Theory



Differential Equation :

$$-\frac{d^2}{dx^2}(b\frac{d^2w}{dx^2}) = f(x)$$

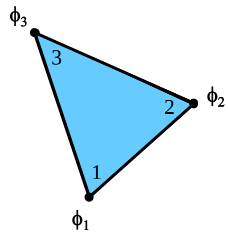
Boundary Condtions Specification :

$$w \ , \ \frac{dw}{dx} \ , \ b\frac{d^2w}{dx^2} \ , \ \frac{d}{dx}(b\frac{d^2w}{dx^2})$$

Two Dimensional Examples

Triangular Element

Scalar-Valued, Two-Dimensional Field Problems



Example Differential Equation :

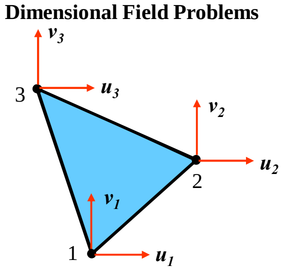
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=f(x,y)$$

Boundary Condtions Specification :

$$\phi\ ,\ \frac{d\phi}{dn}=\frac{\partial\phi}{\partial x}n_x+\frac{\partial\phi}{\partial y}n_y$$

Triangular Element

Vector/Tensor-Valued, Two-Dimensional Field Problems



Elasticity Field Equations in Terms of Displacements

$$\mu\nabla^2 u+\frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x=0$$

$$\mu\nabla^2 v+\frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y=0$$

Boundary Conditons

$$T_x=\left(C_{11}\frac{\partial u}{\partial x}+C_{12}\frac{\partial v}{\partial y}\right)n_x+C_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_y$$

$$T_y=C_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_x+\left(C_{12}\frac{\partial u}{\partial x}+C_{22}\frac{\partial v}{\partial y}\right)n_y$$

Development of Finite Element Equation

- **The Finite Element Equation Must Incorporate the Appropriate Physics of the Problem**
- **For Problems in Structural Solid Mechanics, the Appropriate Physics Comes from Either Strength of Materials or Theory of Elasticity**
- **FEM Equations are Commonly Developed Using *Direct*, *Variational-Virtual Work* or *Weighted Residual* Methods**

Direct Method

Based on physical reasoning and limited to simple cases, this method is worth studying because it enhances physical understanding of the process

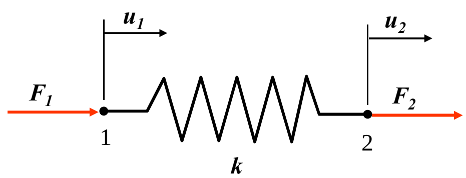
Variational-Virtual Work Method

Based on the concept of virtual displacements, leads to relations between internal and external virtual work and to minimization of system potential energy for equilibrium

Weighted Residual Method

Starting with the governing differential equation, special mathematical operations develop the "weak form" that can be incorporated into a FEM equation. This method is particularly suited for problems that have no variational statement. For stress analysis problems, a Ritz-Galerkin WRM will yield a result identical to that found by variational methods.

Simple Element Equation Example
Direct Stiffness Derivation



Equilibrium at Node 1 $\Rightarrow$ $F_1 = ku_1 - ku_2$

Equilibrium at Node 2 $\Rightarrow$ $F_2 = -ku_1 + ku_2$

or in Matrix Form

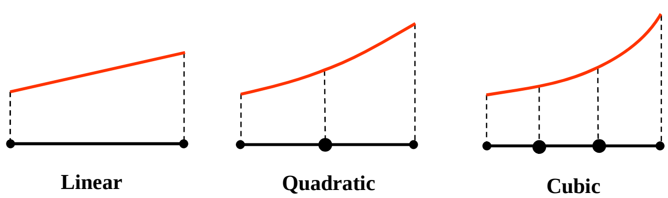
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Stiffness Matrix — $[K]\{u\} = \{F\}$ — **Nodal Force Vector**

Common Approximation Schemes
One-Dimensional Examples

Polynomial Approximation

Most often polynomials are used to construct approximation functions for each element. Depending on the order of approximation, different numbers of element parameters are needed to construct the appropriate function.



Special Approximation

For some cases (e.g. infinite elements, crack or other singular elements) the approximation function is chosen to have special properties as determined from theoretical considerations

One-Dimensional Bar Element

Approximation : $u = \sum_k \psi_k(x) u_k = [\boldsymbol{N}]\{\boldsymbol{d}\}$

Strain : $e = \frac{du}{dx} = \sum_k \frac{d}{dx}\psi_k(x) u_k = \frac{d[\boldsymbol{N}]}{dx}\{\boldsymbol{d}\} = [\boldsymbol{B}]\{\boldsymbol{d}\}$

Stress - Strain Law : $\sigma = Ee = E[\boldsymbol{B}]\{\boldsymbol{d}\}$

$$\int_\Omega \sigma \delta e dV = P_i u_i + P_j u_j + \int_\Omega f \delta u dV \Rightarrow$$

$$\{\boldsymbol{\delta d}\}^T \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}]dx\{\boldsymbol{d}\} = \{\boldsymbol{\delta d}\}^T \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} + \{\boldsymbol{\delta d}\}^T \int_0^L A[\boldsymbol{N}]^T f dx \Rightarrow$$

$$\int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}]dx\{\boldsymbol{d}\} = \{\boldsymbol{P}\} + \int_0^L A[\boldsymbol{N}]^T f dx$$

$$[\boldsymbol{K}]\{\boldsymbol{d}\} = \{\boldsymbol{F}\}$$

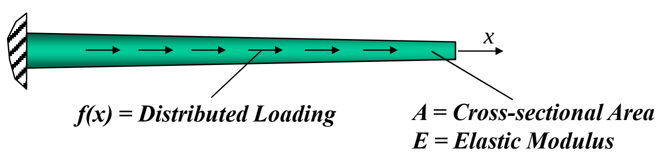
$[K] = \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}]dx$ = Stiffness Matrix

$\{\boldsymbol{F}\} = \begin{Bmatrix} P_i \\ P_j \end{Bmatrix} + \int_0^L A[\boldsymbol{N}]^T f dx$ = Loading Vector

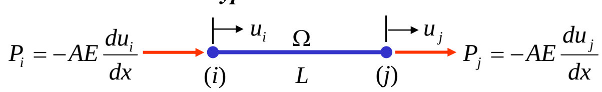
$\{\boldsymbol{d}\} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$ = Nodal Displacement Vector

One-Dimensional Bar Element

Axial Deformation of an Elastic Bar



Typical Bar Element



(Two Degrees of Freedom)

Virtual Strain Energy = Virtual Work Done by Surface and Body Forces

$$\int_V \sigma_{ij}\delta e_{ij} dV = \int_{S_t} T_i^n \delta u_i dS + \int_V F_i \delta u_i dV$$

For One-Dimensional Case

$$\int_\Omega \sigma \delta e dV = P_i u_i + P_j u_j + \int_\Omega f \delta u dV$$

Element Equation

Linear Approximation Scheme, Constant Properties

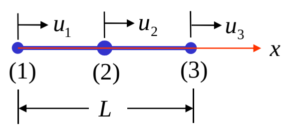
$$[K] = \int_0^L A[\boldsymbol{B}]^T E[\boldsymbol{B}]dx = AE[\boldsymbol{B}]^T[\boldsymbol{B}]\int_0^L dx = AE\begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix}\begin{Bmatrix} -\frac{1}{L} & \frac{1}{L} \end{Bmatrix} L = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

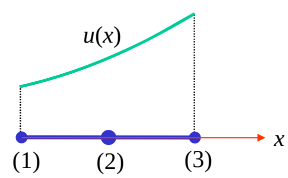
$$\{\boldsymbol{F}\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + \int_0^L A[\boldsymbol{N}]^T f dx = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + Af_o \int_0^L \begin{Bmatrix} -\frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} dx = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + \frac{Af_o L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{\boldsymbol{d}\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \text{Nodal Displacement Vector}$$

$$[\boldsymbol{K}]\{\boldsymbol{d}\} = \{\boldsymbol{F}\} \Rightarrow \frac{AE}{L}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} + \frac{Af_o L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Quadratic Approximation Scheme



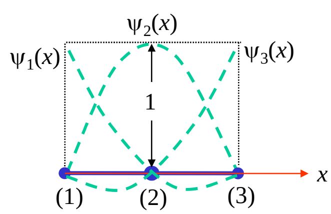


Approximate Elastic Displacement

$$u = a_1 + a_2 x + a_3 x^2 \quad \Rightarrow \quad \begin{aligned} u_1 &= a_1 \\ u_2 &= a_1 + a_2\frac{L}{2} + a_3\frac{L^2}{4} \\ u_3 &= a_1 + a_2 L + a_3 L^2 \end{aligned}$$

$$u = \psi_1(x)u_1 + \psi_2(x)u_2 + \psi_3(x)u_3$$

$$u = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N]\{\boldsymbol{d}\}$$



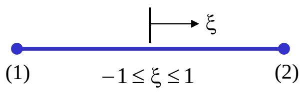
Element Equation

$$\frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Lagrange Interpolation Functions

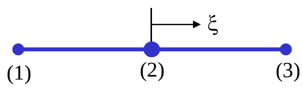
Using Natural or Normalized Coordinates

$$\psi_i(\xi_j) = \begin{cases} 1\,,\ i = j \\ 0\,,\ i \neq j \end{cases}$$



$$\psi_1 = \frac{1}{2}(1-\xi)$$

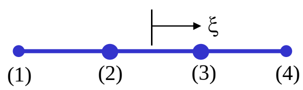
$$\psi_2 = \frac{1}{2}(1+\xi)$$



$$\psi_1 = -\frac{1}{2}\xi(1-\xi)$$

$$\psi_2 = (1-\xi)(1+\xi)$$

$$\psi_3 = \frac{1}{2}\xi(1+\xi)$$



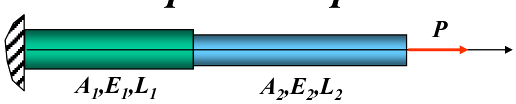
$$\psi_1 = -\frac{9}{16}(1-\xi)(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)$$

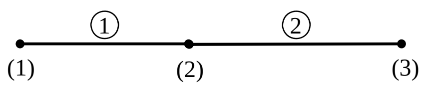
$$\psi_2 = \frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}-\xi)$$

$$\psi_3 = \frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}+\xi)$$

$$\psi_4 = -\frac{9}{16}(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)(1+\xi)$$

Simple Example





Take Zero Distributed Loading

$f = 0$

Global Equation Element 1

$$\frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ 0 \end{Bmatrix}$$

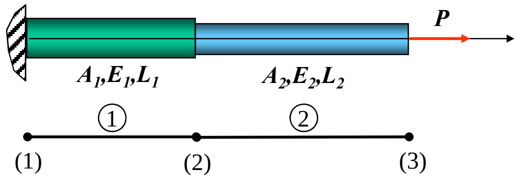
Global Equation Element 2

$$\frac{A_2E_2}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix}$$

Assembled Global System Equation

$$\begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} & 0 \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1} + \frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ 0 & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} + P_1^{(2)} \\ P_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Simple Example Continued



Boundary Conditions

$$U_1 = 0$$
$$P_2^{(2)} = P$$
$$P_2^{(1)} + P_1^{(2)} = 0$$

Reduced Global System Equation

$$\begin{bmatrix} \frac{A_1E_1}{L_1} & -\frac{A_1E_1}{L_1} & 0 \\ -\frac{A_1E_1}{L_1} & \frac{A_1E_1}{L_1}+\frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ 0 & -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} P_1^{(1)} \\ 0 \\ P \end{Bmatrix}$$

$$\begin{bmatrix} \frac{A_1E_1}{L_1}+\frac{A_2E_2}{L_2} & -\frac{A_2E_2}{L_2} \\ -\frac{A_2E_2}{L_2} & \frac{A_2E_2}{L_2} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

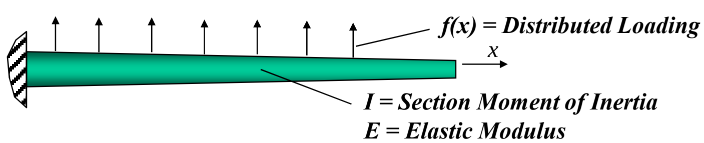
For Uniform Properties A, E, L

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

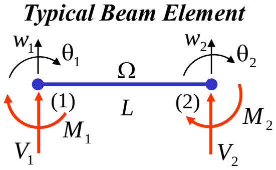
$$\text{Solving} \Rightarrow U_2 = \frac{PL}{AE}\,,\; U_3 = \frac{2PL}{AE}\,,\; P_1^{(1)} = -P$$

One-Dimensional Beam Element

Deflection of an Elastic Beam



Typical Beam Element



(Four Degrees of Freedom)

$$Q_1 = \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)_1 \,,\; Q_2 = \left(EI\frac{d^2w}{dx^2}\right)_1$$

$$Q_3 = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)_2 \,,\; Q_4 = -\left(EI\frac{d^2w}{dx^2}\right)_2$$

$$u_1 = w_1 \,,\; u_2 = \theta_1 = -\left.\frac{dw}{dx}\right|_1 \,,\; u_3 = w_2 \,,\; u_4 = \theta_2 = -\left.\frac{dw}{dx}\right|_2$$

Virtual Strain Energy = Virtual Work Done by Surface and Body Forces

$$\int_\Omega \sigma\delta e dV = Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4w_4 + \int_\Omega f\delta w dV \Rightarrow$$

$$EI\int_0^L [\boldsymbol{B}]^T[\boldsymbol{B}]dx\{\boldsymbol{d}\} = Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4w_4 + \int_0^L f[\boldsymbol{N}]^T dV$$

Beam Approximation Functions

To approximate deflection and slope at each node requires approximation of the form

$$w(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

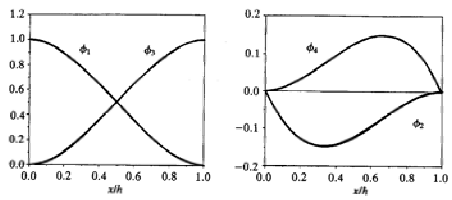
Evaluating deflection and slope at each node allows the determination of c_i thus leading to

$$w(x) = \phi_1(x)u_1 + \phi_2(x)u_2 + \phi_3(x)u_3 + \phi_4(x)u_4 \ ,$$

where ϕ_i are the Hermite Cubic Approximation Functions

$$\phi_1^e = 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3, \quad \phi_2^e = -\bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3, \quad \phi_4^e = -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \frac{\bar{x}}{h_e}\right]$$



Beam Element Equation

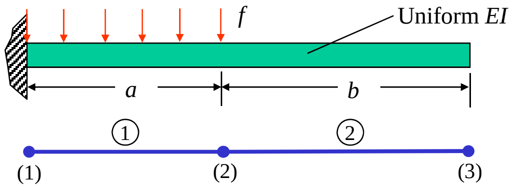
$$EI\int_0^L [\boldsymbol{B}]^T[\boldsymbol{B}]dx\{\boldsymbol{d}\} = Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4w_4 + \int_0^L f[\boldsymbol{N}]^T dV$$

$$\{\boldsymbol{d}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad [\boldsymbol{B}] = \frac{d[\boldsymbol{N}]}{dx} = [\frac{d\phi_1}{dx}\ \frac{d\phi_2}{dx}\ \frac{d\phi_3}{dx}\ \frac{d\phi_4}{dx}]$$

$$[\boldsymbol{K}] = EI\int_0^L [\boldsymbol{B}]^T[\boldsymbol{B}]dx = \frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} \qquad \int_0^L f[\boldsymbol{N}]^T dx = f\int_0^L \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} dx = \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}$$

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}$$

FEA Beam Problem



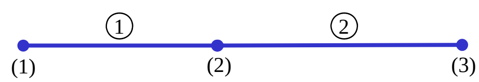
Element 1

$$2EI\begin{bmatrix} 6/a^3 & -3/a^2 & -6/a^3 & -3/a^2 & 0 & 0 \\ -3/a^2 & 2/a & 3/a^2 & 1/a & 0 & 0 \\ -6/a^3 & 3/a^2 & 6/a^3 & 3/a^2 & 0 & 0 \\ -3/a^2 & 1/a & 3/a^2 & 2/a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = -\frac{fa}{12}\begin{Bmatrix} 6 \\ -a \\ 6 \\ a \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \\ Q_4^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

Element 2

$$2EI\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6/b^3 & -3/b^2 & -6/b^3 & -3/b^2 \\ 0 & 0 & -3/b^2 & 2/b & 3/b^2 & 1/b \\ 0 & 0 & -6/b^3 & 3/b^2 & 6/b^3 & 3/b^2 \\ 0 & 0 & -3/b^2 & 1/b & 3/b^2 & 2/b \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_1^{(2)} \\ Q_2^{(2)} \\ Q_3^{(2)} \\ Q_4^{(2)} \end{Bmatrix}$$

FEA Beam Problem



Global Assembled System

$$2EI\begin{bmatrix} 6/a^3 & -3/a^2 & -6/a^3 & -3/a^2 & 0 & 0 \\ \cdot & 2/a & 3/a^2 & 1/a & 0 & 0 \\ \cdot & \cdot & 6/a^3+6/b^3 & 3/a^2-3/b^2 & -6/a^3 & -3/a^2 \\ \cdot & \cdot & \cdot & 2/a+2/b & 3/a^2 & 1/a \\ \cdot & \cdot & \cdot & \cdot & 6/a^3 & 3/a^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2/a \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = -\frac{fa}{12}\begin{Bmatrix} 6 \\ -a \\ 6 \\ a \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)}+Q_1^{(2)} \\ Q_4^{(1)}+Q_2^{(2)} \\ Q_3^{(2)} \\ Q_4^{(2)} \end{Bmatrix}$$

Boundary Conditions

$$U_1 = w_1^{(1)} = 0 \;,\; U_2 = \theta_1^{(1)} = 0 \;,\; Q_3^{(2)} = Q_4^{(2)} = 0$$

Matching Conditions

$$Q_3^{(1)} + Q_1^{(2)} = 0 \;,\; Q_4^{(1)} + Q_2^{(2)} = 0$$

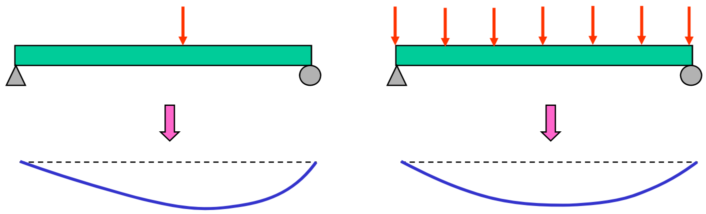
Reduced System

$$2EI\begin{bmatrix} 6/a^3+6/b^3 & 3/a^2-3/b^2 & -6/a^3 & -3/a^3 \\ \cdot & 2/a+2/b & 3/a^2 & 1/a \\ \cdot & \cdot & 6/a^3 & 3/a^2 \\ \cdot & \cdot & \cdot & 2/a \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\frac{fa}{12}\begin{Bmatrix} 6 \\ a \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solve System for Primary Unknowns U_1, U_2, U_3, U_4

Nodal Forces Q_1 and Q_2 Can Then Be Determined

Special Features of Beam FEA



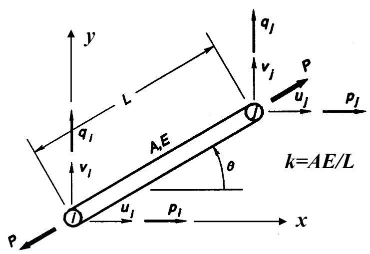
Analytical Solution Gives
Cubic Deflection Curve

Analytical Solution Gives
Quartic Deflection Curve

FEA Using Hermit Cubic Interpolation
Will Yield Results That Match Exactly
With Cubic Analytical Solutions

Truss Element

Generalization of Bar Element With Arbitrary Orientation



Basic Element Equation ($\theta = 0$ case)

$$\begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{Bmatrix} -p_i \\ -q_i \\ -p_j \\ -q_j \end{Bmatrix}$$

Transformation for General Orientation

$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$\{d\} = [T]\{d'\}$ $\quad \{f\} = [T]\{f'\}$

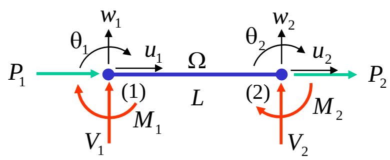
$[k]\{d\} = \{f\} \Rightarrow [T]^T[k][T]\{d'\} = \{f'\}$

$s = \sin\theta$, $c = \cos\theta$

$$[k'] = [T]^T[k][T] = k \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Frame Element

Generalization of Bar and Beam Element with Arbitrary Orientation



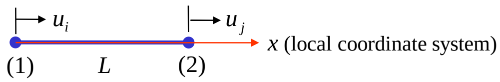
$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ Q_1 \\ Q_2 \\ P_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Element Equation Can Then Be Rotated to Accommodate Arbitrary Orientation

Some Standard FEA References

Bathe, K.J., **Finite Element Procedures in Engineering Analysis**, Prentice-Hall, 1982, 1995.
Beer, G. and Watson, J.O., **Introduction to Finite and Boundary Element Methods for Engineers,** John Wiley, 1993
Bickford, W.B., **A First Course in the Finite Element Method**, Irwin, 1990.
Burnett, D.S., **Finite Element Analysis**, Addison-Wesley, 1987.
Chandrupatla, T.R. and Belegundu, A.D., **Introduction to Finite Elements in Engineering**, Prentice-Hall, 2002.
Cook, R.D., Malkus, D.S. and Plesha, M.E., **Concepts and Applications of Finite Element Analysis**, 3rd Ed., John Wiley, 1989.
Desai, C.S., **Elementary Finite Element Method**, Prentice-Hall, 1979.
Fung, Y.C. and Tong, P., **Classical and Computational Solid Mechanics**, World Scientific, 2001.
Grandin, H., **Fundamentals of the Finite Element Method**, Macmillan, 1986.
Huebner, K.H., Thorton, E.A. and Byrom, T.G., **The Finite Element Method for Engineers**, 3rd Ed., John Wiley, 1994.
Knight, C.E., **The Finite Element Method in Mechanical Design**, PWS-KENT, 1993.
Logan, D.L., **A First Course in the Finite Element Method**, 2nd Ed., PWS Engineering, 1992.
Moaveni, S., **Finite Element Analysis – Theory and Application with ANSYS,** 2nd Ed., Pearson Education, 2003.
Pepper, D.W. and Heinrich, J.C., **The Finite Element Method: Basic Concepts and Applications**, Hemisphere, 1992.
Pao, Y.C., **A First Course in Finite Element Analysis**, Allyn and Bacon, 1986.
Rao, S.S., **Finite Element Method in Engineering**, 3rd Ed., Butterworth-Heinemann, 1998.
Reddy, J.N., **An Introduction to the Finite Element Method**, McGraw-Hill, 1993.
Ross, C.T.F., **Finite Element Methods in Engineering Science,** Prentice-Hall, 1993.
Stasa, F.L., **Applied Finite Element Analysis for Engineers**, Holt, Rinehart and Winston, 1985.
Zienkiewicz, O.C. and Taylor, R.L., **The Finite Element Method**, Fourth Edition, McGraw-Hill, 1977, 1989.

Linear Approximation Scheme



Approximate Elastic Displacement

$$u = a_1 + a_2 x \quad \Rightarrow \quad \begin{matrix} u_1 = a_1 \\ u_2 = a_1 + a_2 L \end{matrix}$$

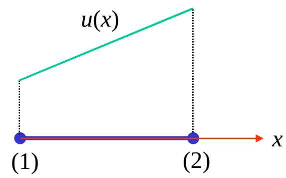
$$\Rightarrow u = u_1 + \frac{u_2 - u_1}{L} x = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2$$

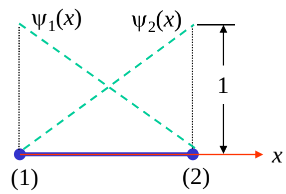
$$= \psi_1(x) u_1 + \psi_2(x) u_2$$

$$u = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{d}\}$$

$[\boldsymbol{N}]$ =Approximation Function Matrix

$\{\boldsymbol{d}\}$ = Nodal Displacement Vector





$\psi_k(x)$ – Lagrange Interpolation Functions