LECTURE NOTE
(MTS 102)

2011/2012 SESSION

UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA

DEPARTMENT OF MATHEMATICS

Course Code	**MTS 102**
Course Title	**CALCULUS AND TRIGONOMETRY**
Number of Units	**3 units**
Course Duration	**3 Hours per week for 3 Weeks**
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Course Outline:

The derivative as a limit of rate of change. Techniques of Differentiation of elementary functions. Applications of derivatives to errors and approximation, Minima and Maxima, Curve sketching.

Prerequsite: Good WAEC/NECO result in Mathematics

Textbooks

1. Robert A. Adams; Calculus, 4th Edition, Addison-Wesley Longman Ltd., Ontario, Canada

What is expected of the Student:

Students are expected to attend all lectures and complete all assignments and examinations. No aids are permitted in examinations.

Evaluation of Student Performance:

1. Midsemester Examination: 20% (Date and lenght to be determined).
2. Written Assignments: 10% (Dates to be announced).
3. Final Examination: 70% (date and time to be determined and fixed by TIMTEC).

0.1 THE DERIVATIVE

Calculus is the mathematics of change , and the primary tool for studying change is a procedure called **Differentiation**.

Rate of Change and Slope

A straight line $y = mx + b$ has the property that its slope is the same at all points. For any other graph, however, the slope may vary from point to point. Thus the slope of the graph of $y = f(x)$ at the point x is itself a function of x. At any point x where the graph has a finite slope we say that f is differentiable and we call the slope the **derivative** of f.
The derivative of a function is another function $f^{'}$ defined by

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e. is a finite real number). If $f^{'}(x)$ exists, we say f is differentiable at x.

REMARK: The value of the derivative of f at a particular point x_o can be expressed as a limit in either of two ways

$$f^{'}(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to xo} \frac{f(x) - f(x_0)}{x - x_0}$$

In the second limit x_oth is replaced by x, so that $h = x - x_0$ and $h \to 0$ is equivalent to $x \to x_0$.
The process of calculating the derivative $f^{'}$ of a given function f is called **differentiation.**

Example: Use the definition of the derivative to calculate the derivatives of the functions

(a) $f(x) = ax + b$
(b) $f(x) = x^2$
(c) $g(x) = \frac{1}{x}$

Solution:
(a.) By the definition of the derivative

$$\begin{aligned} f^{'}(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} \\ f^{'}(x) &= a \end{aligned}$$

(b.) By the definition of the derivative

$$\begin{aligned} f^{'}(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ f^{'}(x) &= 2x \end{aligned}$$

(c.) By the definition of the derivative

$$\begin{aligned} g^{'}(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h} \\ &= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x} \\ &= \lim_{h \to 0} -\frac{1}{x(x+h)} \\ g^{'}(x) &= -\frac{1}{x^2} \end{aligned}$$

Derivative Notation:

The derivative $f^{'}(x)$ of $y = f(x)$ is sometimes written as $\frac{dy}{dx}$ or $\frac{df}{dx}$ (read as "dee y,dee x" or "dee f, dee x"). In this notation, the value of the derivative at $x = c$ [i.e $f'(c)$] is written as

$$\frac{dy}{dx}\mid_{x=c} \qquad \text{or} \qquad \frac{df}{dx}\mid_{x=c}$$

.

For Example , if $y = x^2$, then $\frac{dy}{dx} = 2x$ and the value of this derivative at $x = -3$ is

$$\frac{dy}{dx}\mid_{x=-3} \qquad = 2x\mid_{x=-3} \qquad = 2(-3) = -6$$

The $\frac{dy}{dx}$ notation for derivative suggests slope, $\frac{\Delta y}{\Delta x}$, and can also be thought of as " the rate of change of y with respect to x". Sometimes it is convenient to condense a statement such as

"when $y = x^2$, then $\frac{dy}{dx} = 2x$"

by writing simply

$$\frac{d}{dx}(x^2)$$

by writing simply $\frac{d}{dx}(x^2) = 2x$. Which reads " the derivative of x with respect to x is $2x$"

Techniques of Differentiation

If we had to use the limit definition every time we want to compute a derivative, it would be both tedious and difficult to use calculus in application. In this section, we develop the techniques that greatly simplify the process of differentiation.

The Constant Rule
For any constant c

$$\frac{d}{dx}[c] = 0$$

i.e the derivative of a constant is zero

The Power Rule
For any real number n

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example:

1. $\frac{d}{dx}[x^7] = 7x^{7-1} = 7x^6$.
2. $\frac{d}{dx}[\sqrt[3]{x^2}] = \frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$.
3. $\frac{d}{dx}(\frac{1}{x^5}) = \frac{d}{dx}(x^{-5}) = -5x^{-5-1} = -5x^{-6}$.

The Constant Multiple Rule
If c is a constant and $f(x)$ is differentiable, then so is $cf(x)$ and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example:

1. $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$
2. $\frac{d}{dx}(\frac{-7}{\sqrt{x}}) = \frac{d}{dx}(-7x^{-\frac{1}{2}}) = -7(-\frac{1}{2}x^{\frac{-3}{2}}) = \frac{7}{2}x^{\frac{-3}{2}}$

The Sum Rule
If $f(x)$ and $g(x)$ are differentiable, then so is the sum $S(x) = f(x) + g(x)$ and $S^{'}(x) = f^{'}(x) + g^{'}(x)$; that is

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Example: Differentiate the following polynomials

1. $y = 5x^3 - 4x^2 + 12x - 8$
2. $y = 2x^5 - 3x^{-7}$

Solution:
(1) Differentiating y we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[5x^3 - 4x^2 + 12x - 8] \\
&= \frac{d}{dx}[5x^3] + \frac{d}{dx}[-4x^2] + \frac{d}{dx}[12x] + \frac{d}{dx}[-8] \\
&= 15x^2 - 8x^1 + 12x^0 + 0 \\
&= 15x^2 - 8x + 12
\end{aligned}$$

(2) Differentiating y we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[2x^5 - 3x^{-7}] \\ &= \frac{d}{dx}[2x^5] + \frac{d}{dx}[-3x^{-7}] \\ &= 10x^4 + 21x^{-8}\end{aligned}$$

The Product Rule
If $f(x)$ and $g(x)$ are differentiable, then so is their product $P(x) = f(x)g(x)$ and that is

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

or equivalently $(fg)^{'} = fg^{'} + gf^{'}$

Example of Product Rule
(a) Find $\frac{dy}{dx}$ if $y = (2\sqrt{x} + \frac{3}{x})(3\sqrt{x} - \frac{2}{x})$

Solution: Applying the Product Rule with $f(x)$ and $g(x)$ being the two functions enclosed in the large parenthesis, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[\left(2\sqrt{x} + \frac{3}{x}\right)\left(3\sqrt{x} - \frac{2}{x}\right)\right] \\ &= \left(\frac{1}{\sqrt{x}} - \frac{3}{x^2}\right)\left(3\sqrt{x} - \frac{2}{x}\right) + \left(2\sqrt{x} + \frac{3}{x}\right)\left(\frac{3}{2\sqrt{x}} + \frac{2}{x^2}\right) \\ &= 6 - \frac{5}{2x^{\frac{3}{2}}} + \frac{12}{x^3}\end{aligned}$$

(b) Let $y = uv$ be the product of the functions u and v. Find $y^{'}(2)$ if $u(2) = 2$, $u^{'}(2) = -5$, $v(2) = 1$, and $v^{'}(2) = 3$

Solution: From the product Rule we have

$$y^{'} = (uv)^{'} = u^{'}v + uv^{'}$$

Therefore

$$\begin{aligned}y^{'}(2) &= u^{'}(2)v(2) + u(2)v^{'}(2) \\ &= (-5)(1) + (2)(3) \\ &= -5 + 6 \\ &= 1\end{aligned}$$

The product rule can be extended to products of any number of factors, for instance

$$\begin{aligned}(fgh)'(x) &= f'(x)gh(x) + f(x)(gh)'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)\end{aligned}$$

In general, the derivative of a product of n functions will have n terms; each term will be the same product but with one of the factors replaced by its derivatives

$$(f_1 f_2 f_3 \cdots f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 f_3 \cdots f_n'$$

The Reciprocal Rule
If $f(x)$ is differentiable and $f(x) \neq 0$, then $\frac{1}{f(x)}$ is differentiable and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}$$

Example: Differentiate the function

1. $y = \frac{1}{x^2+1}$
2. $f(t) = \frac{1}{t+\frac{1}{t}}$

Solution:
1. Using the Reciprocal Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2+1}\right) = \frac{2x}{(x^2+1)^2}$$

2. Using the Reciprocal Rule

$$\begin{aligned}f'(t) &= \frac{-1}{(t+\frac{1}{t})^2}\left(1 - \frac{1}{t^2}\right) \\ &= \frac{-t^2}{(t^2+1)^2}\left(\frac{t^2-1}{t^2}\right) \\ &= \frac{1-t^2}{(t^2+1)^2}\end{aligned}$$

The Quotient Rule
The Product Rule and the Reciprocal Rule can be combined to provide a

rule for differentiating a quotient of two functions. Observe that

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x).\frac{1}{g(x)}\right)\\ &= f^{'}(x)\frac{1}{g(x)}+f(x)\left(\frac{-g(x)}{(g(x))^2}\right)\\ &= \frac{g(x)f^{'}(x)-f(x)g^{'}(x)}{(g(x))^2}\end{aligned}$$

If $f(x)$ and $g(x)$ are differentiable at x, and if $g(x)\neq 0$ then the quotient $\frac{f(x)}{g(x)}$ is differentiable at x and

$$\left(\frac{f}{g}\right)^{'}(x)=\frac{g(x)f^{'}(x)-f(x)g^{'}(x)}{(g(x))^2}$$

Example: Find the derivatives of

1. $y=\frac{1-x^2}{1+x^2}$
2. $f(\theta)=\frac{a+b\theta}{m+n\theta}$

Solution:
a.) Using Quotient rule we have:

$$\frac{dy}{dx}=\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}=-\frac{4x}{(1+x^2)^2}$$

b.) Using Quotient rule we have:

$$f^{'}(\theta)=\frac{(m+n\theta)(b)-(a+b\theta)(n)}{(m+n\theta)^2}=\frac{mb-na}{(m+n\theta)^2}$$

The Chain Rule
If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is in turn a differentiable function of x, then the composite function $y=f(g(x))$ is a differentiable function of x whose derivative is given by the product

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

or equivalently by

$$\frac{dy}{dx}=f^{'}(g(x))g^{'}(x)$$

Example: a.) Find $\frac{dy}{dx}$ if $y = (x^2+2)^3 - 3(x^2+2)^2 - 1$

Solution: Note that $y = u^3 - 3u^2 + 1$, where $u = (x^2+2)$.
Thus $\frac{dy}{dx} = 3u^2 - 6u$ and $\frac{du}{dx} = 2x$ and according to the chain rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (3u^2 - 6u)(2x) \\ &= [3(x^2+2)^2 - 6(x^2+2)](2x) \\ &= 6x(x^2+2)[(x^2+2) - 2] \\ &= 6x(x^2+2)x^2 \\ &= 6x^3(x^2+2) \end{aligned}$$

b.) Consider the function $y\frac{u}{u+1}$, where $u = 3x^2 - 1$

$$\begin{aligned} \frac{dy}{du} &= \frac{(u+1)(1) - u(1)}{(u+1)^2} \\ &= \frac{1}{(u+1)^2} \end{aligned}$$

and

$$\frac{du}{dx} = 6x$$

According to the chain rule, it follows that

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left[\frac{1}{(u+1)^2}\right] 6x \\ &= \frac{6x}{(u+1)^2} \\ &= \frac{6x}{(3x^2 - 1 + 1)^2} \\ &= \frac{6x}{(3x^2)^2} \\ &= \frac{2}{x^3} \\ &= 2x^{-3}. \end{aligned}$$

Derivatives Of Trigonometric Functions

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\cot x) &= \csc^2 x \\
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= \sec x \tan x \\
\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\
&= -\csc x \cot x
\end{aligned}$$

Example: Differentiate the following.

1. $y = 3x + \cot\left(\frac{x}{2}\right)$
2. $y = \frac{3}{\sin 2x}$

Solution:
1.) Differentiating $y = 3x + \cot\left(\frac{x}{2}\right)$ we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[3x + \cot\left(\frac{x}{2}\right)\right] \\
&= 3 + \frac{1}{2}\left[\csc^2\left(\frac{x}{2}\right)\right] \\
&= 3 - \frac{1}{2}\csc^2\left(\frac{x}{2}\right)
\end{aligned}$$

2.) Differentiating $y = \frac{3}{\sin 2x}$ we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{3}{\sin 2x}\right] \\
&= \frac{d}{dx}[3\csc(2x)] \\
&= (2)3(-\csc(2x)\cot(2x)) \\
&= -6\csc(2x)\cot(2x)
\end{aligned}$$

Differentiation of Exponential Function
The derivative of the exponential function

$$\frac{d}{dx}(e^x) = e^x$$

for every real number x. If $u(x)$ is a differentiable function of x, then

$$\frac{d}{dx}(e^{u(x)}) = e^{u(x)}\frac{du}{dx}$$

Example: Differentiate the function $f(x) = e^{x^2+1}$.

Solution: Using the Chain rule with $u = x^2 + 1$, we find

$$f^{'}(x) = e^{x^2+1}\left[\frac{d}{dx}(x^2+1)\right] = 2xe^{x^2+1}$$

Differentiation of Logarithm Function
The derivative of $\ln x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

for $x > 0$. If $u(x)$ is differentiable function of x, then

$$\frac{d}{dx}[\ln u(x)] = \frac{1}{u(x)}\frac{du}{dx}$$

Example: Differentiate the function $f(x) = \ln(2x^3 + 1)$.

Solution: Let $f(x) = \ln u$, where $u(x) = 2x^3 + 1$

$$\begin{aligned} f^{'}(x) &= \frac{1}{u}\frac{du}{dx} \\ &= \frac{1}{2x^3+1}\frac{d}{dx}(2x^3+1) \\ &= 2\frac{3x^2}{2x^3+1} \\ &= \frac{6x^2}{2x^3+1} \end{aligned}$$

Logarithmic Differentiation
Sometimes you can simplify the work involved in differentiating a function if you first take its logarithm. This techniques is called Logarithmic differentiation.

Example: Differentiate the function

$$f(x) = \frac{\sqrt[3]{x+1}}{(1-3x)^4}$$

Solution: Quotient rule and Chain rule could be used but the resulting computation could be tedious. A more efficient approach is to take the logarithm of both sides of the expression of f

$$\begin{aligned}
\ln f(x) &= \ln\left[\frac{\sqrt[3]{x+1}}{(1-3x)^4}\right] \\
&= \ln\sqrt[3]{x+1} - \ln(1-3x)^4 \\
\ln f(x) &= \frac{1}{3}\ln(x+1) - 4\ln(1-3x)
\end{aligned}$$

Using chain rule for logarithm to differentiate both side.

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= \frac{1}{3}\left(\frac{1}{x+1}\right) - 4\left(\frac{-3}{1-3x}\right) \\
&= \frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{12}{1-3x} \\
f'(x) &= f(x)\frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{12}{1-3x} \\
f'(x) &= \left[\frac{\sqrt[3]{x+1}}{(1-3x)^4}\right]\left[\frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{12}{1-3x}\right]
\end{aligned}$$

Implicit Differentiation
The functions of the form $y = f(x)$ in which the dependent variable y on the left is given explicitly by an expression on the right involving the independent variable x. A function in this form is said to be in explicit form.
Example $y = x^2 + 3x + 1$, $y = \sqrt{1+x^2}$

But equation such as $x^2y^3 = 5y^3 + x$, $x^2y + 2y^3 = 3x + 2y$ is said to define y implicitly as a function of x and in such function y is said to be implicit form.

To compute the $\frac{dy}{dx}$ of this function, we use a simple techniques based on the chain rule that you can use to find $\frac{dy}{dx}$ without first solving for y explicitly. This technique, known as Implicit differentiation, consists of differentiating both sides of given equation with respect to x and then solving algebraically for $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ if $x^2y + y^2 = x^3$
Solution: Let $y = f(x)$

$$\begin{aligned}
\frac{d}{dx}[x^2 f(x) + (f(x))^2] &= \frac{d}{dx}(x^3) \\
x^2\frac{df}{dx} + f(x)\frac{d}{dx}(x^2) + 2f(x)\frac{df}{dx} &= 3x^2 \\
x^2\frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} &= 3x^2 \\
x^2\frac{df}{dx} + 2f(x)\frac{df}{dx} &= 3x^2 - 2xf(x)
\end{aligned}$$

x$^2 + 2f(x)\frac{df}{dx} = 3x^2 - 2xf(x)$

$$\frac{df}{dx} = \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)}$$

Finally, replace $f(x)$ by y to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}$$

Maximal and Minimal

Increasing and Decreasing Function: Let $f(x)$ be a function defined on the interval $a < x < b$, and let x_1 and x_2 be two numbers in the interval. Then f(x) is increasing on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$. $f(x)$ is decreasing on the interval if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

(a.) $f(x)$ is increasing on $a < x < b$

(b.) $f(x)$ is decreasing on $a < x < b$
A function is said to be increasing when $f^{'}(x) > 0$ and it's said to be decreasing when $f^{'}(x) < 0$.
The graph of the function $f(x)$ is said to have a relative maximum at $x = c$ if $f(c) \geq f(x)$ for all x in an interval $a < x < b$ containing c. Similarly, the graph has a relative minimum at $x = c$ if $f(c) \leq f(x)$ on such an interval. Collectively, the relative maximal and minimal of f are called
Relative Extrema. A number c in the domain of $f(x)$ is called a critical number if either $f^{'}(c) = 0$ or $f'(c)$ does not exist. The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a Critical Point.

Conditions for Maximal and Minimal

1. Let $f(x)$ be maximal at $x = c$. Now just before the maximal value i.e.

at $x = c$, the function is increasing (see Fig a). Therefore,

$$f'(x) = \frac{dy}{dx} = +\text{ve}$$

Just after the maximum value i.e. at $x = c$ the function is decreasing (see fig a). Therefore

$$f'(x) = \frac{dy}{dx} = -\text{ve}$$

Thus in passing through a maximum value at $x = c$ the derivative changes its sign from $+ve$ to $-ve$. Therefore

$f'(x) = \frac{dy}{dx} = 0$ at x = c

2. Again if $y = f(x)$ is maximal at $x = c$ then $\frac{dy}{dx}$ changes from $+ve$ to $-ve$ as it passes through $x = c$. Therefore

   $\frac{dy}{dx}$ is a decreasing function

   So the derivative of $\frac{dy}{dx}$ should be negative at $x = c$ or

$$\frac{d^2y}{dx^2} = -\text{ve}$$

   at $x = c$

Here are the two conditions for maximal

1. $\frac{dy}{dx} = 0$
2. $\frac{d^2y}{dx^2} = -\text{ve}$

Similarly conditions for minimal

1. $\frac{dy}{dx} = 0$
2. $\frac{d^2y}{dx^2} = +\text{ve}$

Example
Find the points at which the function $y = x^3 + 6x^2 - 15x + 5$ has maximum and minimum values
Solution

$$y = x^3 + 6x^2 - 15x + 5$$
$$\frac{dy}{dx} = 3x^2 + 12x - 15$$

For maximal and minimal $\frac{dy}{dx} = 0$

$$\begin{aligned} 3x^2 + 12x - 15 &= 0 \\ x^2 + 4x - 5 &= 0 \\ x &= 1 \quad \text{or} \quad 5 \end{aligned}$$

After differentiating (ii) we have

$$f''(x) = \frac{d^2y}{dx^2} = 6x + 12$$

1. When $x = 1$, $\frac{d^2y}{dx^2} = 6x + 12 = +\text{ve}$.
   The given function is minimum at $x = 1$
   Minimum value of the function is

$$\begin{aligned} y &= x^3 + 6x^2 - 15x + 5 \\ &= (1)^3 + 6(1)^2 - 15(1) + 5 \\ &= 1 + 6 - 15 + 5 \\ &= -3 \end{aligned}$$

2. When $x = -5$, $\frac{d^2y}{dx^2} = 6(-5) + 12 = -18 = -\text{ve}$.
   The given function is maximum at $x = -5$.
   Maximum value of the function is

$$\begin{aligned} y &= x^3 + 6x^2 - 15x + 5 \\ &= (-5)^3 + 6(-5)^2 - 15(-5) + 5 \\ &= -125 + 150 + 75 + 5 \\ &= 105 \end{aligned}$$

Curve Sketching:
Limit involving infinity can be used to graphical features called *asymptotes*. In particular, the graph of a function $f(x)$ is said to have a **Vertical Asymptote** at $x = c$ if $f(x)$ increases or decreases without bound as x tends toward c, from either the right or the left.

For instance, consider the rational function

$$f(x) = \frac{x+1}{x-2}$$

As x approaches 2 from the left ($x < 2$), the functional values decreases without bound, but they increase without bound if the approach is from the right ($x > 2$). This behaviour is illustrated in the table and demonstrated graphically below.

x	1.95	1.97	1.99	1.999	2	2.001	2.005	2.01
$f(x) = \frac{x+1}{x-2}$	-59	-99	-299	-2999	undefined	3001	601	301

Vertical Asymptotes The Line $x = c$ is a vertical asymptote of the graph of $f(x)$ if either

$$\lim_{x \to c^-} = +\infty \text{ (or } -\infty)$$

or

$$\lim_{x \to c^+} = +\infty \text{ (or } -\infty)$$

In general, a rational function $R(x) = \frac{p(x)}{q(x)}$ has a vertical asymptotes $x = c$ whenever $q(c) = 0$ but $p(c) \neq 0$.

Horinzontal Asymptotes The horizontal line $y = b$ is called a horizontal asymptote of the graph of $y = f(x)$ if either

$$\lim_{x \to -\infty} = b$$

or

$$\lim_{x \to +\infty} = b$$

A General Procedure for Sketching the Graph of $f(x)$

1. Find the domain of $f(x)$ (that is where f(x) is defined)
2. Find and plot all intercepts. The y intercept (where $x = 0$)and the x intercept (where $f(x) = 0$)

3. Determine all vertical and horizontal asymptotes of the graph. Draw the asymptotes in a coordinate plane.
4. Find $f'(x)$ and use it to determine the critical number of $f(x)$ and intervals of increase and decrease.
5. Determine all relative extrema (both coordinates). Plot each relative maximum with a "cap"($\cap$) and each relative minimum with a "cup"($\cup$)
6. Find $f''(x)$ and use it to determine intervals and points of inflection. Plot each inflection points with a "twist" to suggest the shape of the graph near the point.
7. You now have a preliminary graph, with asymptotes in place,intercepts plotted, arrows indicating the direction of the graph, and "caps", "cups", and "twist" suggesting the shape at key points. Plot additional points if needed, and complete the sketch by joining the plotted points in the directions indicated. Be sure to remember that the graph cross a vertical asymptotes.

EXERCISE 1

Find the derivative of the following functions:

1. $y = x^{-4}$
2. $y = \frac{4}{3}\pi r^3$
3. $y = -\frac{x^2}{16} + \frac{2}{x} - x^{\frac{3}{2}} + \frac{1}{3x^2} + \frac{x}{3}$
4. $y = \frac{x^5 - 4x^2}{x^3}$
5. $g(t) = \frac{t^2 + \sqrt{t}}{2t+5}$
6. $y = \frac{1}{3}(x^5 - 2x^3 + 1)\left(x - \frac{1}{x}\right)$
7. $f(t) = \frac{t^2+2t+1}{t^2+3t-1}$
8. $y = (7x-3)^{10}$
9. $y = \left(3x + \frac{1}{(2x+1)^3}\right)^{\frac{1}{4}}$
10. Find $\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)|_{x=-2}$
11. $y = \sqrt{sinx}$
12. $y = 2e^{3x} + \tan x - \cos 2x + 9sin^{-1}x$
13. $y = x^5 \sin x \cos x$
14. $y = \sin^4 x \cos^3 x$
15. If $x^3y + xy^3 - 3xy^2 = 8$ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 3$, $y = 2$

EXERCISE 2

Differentiate the following Logarithm and Exponential Function.

1. $L(x) = \ln \frac{x^2+2x-3}{x^2+2x-1}$
2. $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
3. $g(s) = (e^s + s + 1)(2e^{-s} + s)$
4. $g(u) = \ln(u + \sqrt{u^2+1})$
5. $y = x^4 e^{3x} \tan x$

6. $y = \ln\left(\frac{e^{3x}}{1+x}\right)$

7. $y = \ln\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$

EXERCISE 3

Find the value of x for which the following function is a maximum, minimum.

1. $y = 9x^3 - 45x^2 + 48x + 11$
2. $y = 11 - 12x + 6x^2 - x^3$
3. $y = (x-2)^3(x-3)^2$
4. $y = 3sin^2x + 4cos^2x$